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Introduction

In 300 B.C. Euclid treated Geometry as a deductive system. In 1637 Rene Du Perron Descartes introduced the coordinate system in Euclidean Geometry. Tensor is a positive outcome of the introduction of coordinate system.

One can well say that tensors came of age with the appearance of the remarkable paper of the famous mathematicians Ricci and Levi-Civita (Methods de Calcul differential absolu et leurs applications, Mathematische Annalen, vol. 54, 1901). But, as is well known, it was Einstein's use of tensors in his general theory of relativity that was mainly responsible for the sudden emergence of the tensor calculus as a popular field of mathematical activity.

Tensor calculus is concerned with a study of abstract objects, called tensors, whose properties are independent of the reference frames used to describe the objects. A tensor is represented in a particular reference frame by a set of functions, termed its components, just as a vector is determined in a given reference frame by a set of components and if a new coordinate system is introduced, the same tensor is determined by a new set of components and the new components are related to the old ones, in a different way (contravariant or covariant way).

Since tensor analysis deals with entities and properties that are independent of the choice of reference frames it forms an ideal tool for the study of natural laws. In particular, Einstein found it an excellent tool for the presentation of his 'General Theory of Relativity'. As a result tensor calculus came into general prominence and is now invaluable in its applications to most branches of Theoretical Physics, it is also indispensable in the Differential Geometry of curves and surfaces in Euclidean space.

Tensor is a generalization of the term 'vector' and tensor calculus is a generalization of vector calculus.

So we first discuss how the term 'tensor' may be considered as a generalization of the term 'vector'.

For this we start with a 3-dimensional Euclidean space E_3 with a rectangular Cartesian coordinate system.

Let B and A be two points of E_3 with coordinates (x^1, x^2, x^3) and (y^1, y^2, y^3) respectively. Then the coordinates of the vector $\overrightarrow{AB}$ are given by $(x^1 - y^1, x^2 - y^2, x^3 - y^3)$.

Let

$$z^i = x^i - y^i, \qquad i = 1, 2, 3. \tag{i}$$

Now we consider an orthogonal transformation in E_3 given by

$$\left.\begin{aligned} \bar{x}^1 &= l_1x^1 + m_1x^2 + n_1x^3 + a_1 \\ \bar{x}^2 &= l_2x^1 + m_2x^2 + n_2x^3 + a_2 \\ \bar{x}^3 &= l_3x^1 + m_3x^2 + n_3x^3 + a_3 \end{aligned}\right\} \quad \text{(ii)}$$

where (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the direction cosines of the transformed axes with respect to the old coordinate system and (a_1, a_2, a_3) are the coordinates of the new origin with respect to the old coordinate system. Then (ii) can be expressed as

$$\bar{x}^i = l_ix^1 + m_ix^2 + n_ix^3 + a_i, \quad i = 1, 2, 3. \quad \text{(iii)}$$

We denote the coordinates of B and A by $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ and $(\bar{y}^1, \bar{y}^2, \bar{y}^3)$ respectively in the new coordinate system.

$$\text{Let,} \quad \bar{z}^i = \bar{x}^i - \bar{y}^i.$$

Then from (iii) we get

$$\begin{aligned} \bar{z}^i &= \bar{x}^i - \bar{y}^i \\ &= (l_ix^1 + m_ix^2 + n_ix^3 + a_i) - (l_iy^1 + m_iy^2 + n_iy^3 + a_i) \\ &= l_iz^1 + m_iz^2 + n_iz^3. \end{aligned} \quad \text{(iv)}$$

From (ii) we get

$$\frac{\partial \bar{x}^1}{\partial x^1} = l_1, \quad \frac{\partial \bar{x}^1}{\partial x^2} = m_1, \quad \frac{\partial \bar{x}^1}{\partial x^3} = n_1, \text{ etc.}$$

Hence from (iv) we get

$$\begin{aligned} \bar{z}^i &= \frac{\partial \bar{x}^i}{\partial x^1}z^1 + \frac{\partial \bar{x}^i}{\partial x^2}z^2 + \frac{\partial \bar{x}^i}{\partial x^3}z^3 \\ &= \sum_{j=1}^{3} \frac{\partial \bar{x}^i}{\partial x^j}z^j, \qquad i = 1, 2, 3. \end{aligned} \quad \text{(v)}$$

The above equations show that the coordinates of a vector $\overline{AB}$ of E_3 transform according to a certain law given by (v), when referred to a new coordinate system. Hence a vector of E_3 may be considered as an object which is determined in a given coordinate system by a set of components which transform according to the law given by (v), when referred to a new coordinate system.

Now we show that there exists other objects which are determined in a given coordinate system by a set of components which transform according to other laws when referred to a new system of coordinates.

Let z^i and w^i be two vectors of E_3 and let an object have z^1w^1, z^1w^2, z^1w^3, z^2w^1, z^2w^2, z^2w^3, z^3w^1, z^3w^2 and z^3w^3 as its components in a rectangular Cartesian coordinate system. Denote the components of z^i and w^i in another system of coordinates by $\bar{z}^i$ and $\bar{w}^i$ respectively. Then by (v) we get

$$\bar{z}^i\bar{w}^j = \sum_{i,j=1}^{3} \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^l}z^kw^l. \quad \text{(vi)}$$

Denote $z^k w^l$ by A^{kl} and $\bar{z}^i \bar{w}^j$ by $\bar{A}^{ij}$, then we can express (vi) as

$$\bar{A}^{ij} = \sum_{i,j=1}^{3} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} A^{kl} \qquad \text{(vii)}$$

Thus we see that there exists an object with components A^{kl} in a certain coordinate system which transform according to the rule given by (vii), when referred to a new system of coordinates.

Similarly, if we consider n vectors we can show that there exists an object with components $C^{j_1 j_2 \cdots j_n}$ in a certain coordinate system which transform according to the law

$$\bar{C}^{i_1 i_2 \cdots i_n} = \sum \frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{j_2}} \cdots \frac{\partial \bar{x}^{i_n}}{\partial x^{j_n}} C^{j_1 j_2 \cdots j_n}. \qquad \text{(viii)}$$

We observe that

I) in case of a vector z^i, only one term $\dfrac{\partial \bar{x}^i}{\partial x^j}$ occurs in each term of the summation,

II) in case of an object with components A^{ij}, two terms $\dfrac{\partial \bar{x}^i}{\partial x^k}$ and $\dfrac{\partial \bar{x}^j}{\partial x^l}$ occur in each term of the summation and

III) in case of an object with components $C^{j_1 j_2 \cdots j_n}$ only n terms $\dfrac{\partial \bar{x}^{i_1}}{\partial x^{j_1}}, \dfrac{\partial \bar{x}^{i_2}}{\partial x^{j_2}}, \ldots, \dfrac{\partial \bar{x}^{i_n}}{\partial x^{j_n}}$ occur in each term of summation.

From the above we conclude that objects with components A^{ij}, $C^{j_1 j_2 \cdots j_n}$ may be included under a general term called 'tensor' and objects with components $A^{ij}, \ldots, C^{j_1 j_2 \cdots j_n}$, z^i are called tensors of rank $2, \ldots, n$ and 1 respectively. Consequently a vector of E_3 may be called a tensor of rank 1. Hence we conclude that a tensor of E_3 may be regarded as a generalization of a vector of E_3 defined from the transformation standpoint.

We also use the term 'order' in place of rank.

Similarly a tensor of E_n can be obtained as a generalization of a vector of E_n. A tensor, obtained from orthogonal transformation of rectangular Cartesian coordinate system is called a Cartesian tensor.

Instead of E_n with rectangular Cartesian coordinate system, if we consider any general space and general transformation in lieu of orthogonal transformation we get a general tensor, which we shall call simply a tensor.

I Some Preliminaries

I.0 Introduction

This chapter is devoted to a discussion of the notations alone, applying it briefly to the theory of determinants, some conventions and symbols, which will be needed later on.

I.1 Systems of Different Orders

If we are given a set of two independent variables, they may be denoted by two different quantities, such as x and y , but it is more convenient to denote these variables by the same letter, distinguishing them by means of indices. Thus we may write the quantities as x_1, x_2 or as they may be more compactly written, as[1]

$$x_i \quad (i = 1, 2)\,. \tag{I.1.1}$$

In (I.1.1) we have written the index i as a subscript, but we can equally well use superscripts instead of subscript, so that the variables can be written as x^1, x^2 or as

$$x^i \quad (i = 1, 2)\,. \tag{I.1.2}$$

Here it must be understood that x^i does not mean that x is raised to the i-th power, but i is used merely to distinguish between the two quantities.

Systems of quantities, which, like x^i and x_i , depend on one index only, are called systems of the first order or simple systems and x^1 , x^2 ; x_1 , x_2 are called the components of the system. It is obvious that the system of the first order has two components if the index take the values 1 and 2.

Again, we consider four quantities which may be written in the following form :

$$a_{11}, a_{12}, a_{21}, a_{22} \tag{I.1.3}$$

$$a^{11}, a^{12}, a^{21}, a^{22} \tag{I.1.4}$$

and

$$a^1_1\ , a^1_2\ , a^2_1\ , a^2_2\,. \tag{I.1.5}$$

Then in indexed notation (I.1.3), (I.1.4) and (I.1.5) may be written as a_{ij}, a^{ij} and a^i_j respectively, where i, j take values 1 and 2. Thus the systems a_{ij}, a^{ij} and a^i_j

[1] From now on the equations are in the form (C.S.E) where C stands for chapter, S for section number and E stands for equation number. Thus (III.2.8) means equation number 8 of the second section of the third chapter and so on.

depends on two indices, are called double systems or systems of second order, whose number of components is 4 *i.e.*, 2^2 for $i, j = 1, 2$.

Similarly we have systems of the third order, which depends on three indices and which may be of four types as $a^{ijk}, a_{ijk}, a^i_{jk}, a^{ij}_k$, $(i, j, k = 1, 2)$ and then the number of components is 8 *i.e.*, 2^3. Thus in general, if the indices of a system take values $1, 2, \ldots, n$, then the number of its components is n, n^2, n^3 according as it is of the first, second, third order respectively. In a similar manner we can obtain a system of the n-th order. We shall call a single quantity a , which has no indices, a system of order zero.

The upper and lower indices of a system are called its indices of contravariance and covariance respectively. Hence for the system a^i_{jk}, the index i is the index of contravariance and the indices j, k are covariance indices. The system a^{ij} consisting upper indices only is called a contravariant system and the system a_{ij} is called a covariant system, whereas the system a^i_j is called a mixed system.

I.2 Summation Convention

If in some expression a certain index occurs twice, once in the upper index and once in the lower index, we shall mean that this expression is summed with respect to that index for all admissible values of the index. We now consider the expression

$$\sum_{i=1}^{n} a_i x^i = a_1 x^1 + a_2 x^2 + \ldots + a_n x^n,$$

where the index i occurs twice in the left hand side, once as the upper index and once as the lower index. Then using summation convention due to A. Einstein, we shall omit the summation symbol $\sum$ and write $a_i x^i$ to mean $a_1 x^1 + a_2 x^2 + \ldots\ldots + a_n x^n$.

Dummy Index : If in an indexed expression an index occurs twice, once as a lower index and once again as an upper index, then that index is called a dummy index. Thus in the expression $a_i x^i$ the index i is dummy while in the expression $a_{ij} x^i x^j$ both the indices i and j are dummy. In the sum $a_i x^i$ the dummy index i may be replaced by any letter *i.e.*, $a_i x^i = a_j x^j$. The use of dummy index is analogous to a variable of integration in a definite integral, which can be changed at will *i.e.*,

$$\int_a^b f(x)dx = \int_a^b f(z)dz.$$

It is to be noted that when two dummy indices occur in an indexed expression then the indices should not be written by the same letter. Hence in the expression $a_{ij} x^i x^j$ the dummy indices i and j can not be replaced by the same letter k *i.e.*, $a_{ij} x^i x^j$ can not be written as $a_{kk} x^k x^k$.

Free index : If in an indexed expression an index is not dummy, *i.e.*, it is not repeated then it is called a free index. Thus in the expression $a_{ij} x^i x^k$, j and k are the free indices while i is the dummy index. It may be noted that when in an expression both dummy and free indices occur, then the dummy index should not be

denoted by the same letter as that denoting the free index. Thus in the expression $a_{ij}x^i x^k$, the dummy index i can not be denoted by the letter j or k which are free indices.

We now restate the summation convention as follows :

When in an indexed expression a dummy index occurs, the expression is to be considered as the sum and is to be summed by giving all the values of the index.

I.3 Kronecker Symbols

A particular system of second order denoted by δ^i_j and defined by

$$\begin{aligned} \delta^i_j &= 1 \quad \text{for} \quad i = j \\ &= 0 \quad \text{for} \quad i \neq j, \quad i, j = 1, 2, \ldots, n \end{aligned}$$

is called the Kronecker symbol or Kronecker delta. We now consider some properties of this system :

1) If $x^1, x^2, \ldots, x^n$ are independent variables, then

$$\begin{aligned} \frac{\partial x^i}{\partial x^j} &= 1 \quad \text{for} \quad i = j \\ &= 0 \quad \text{for} \quad i \neq j, \quad i, j = 1, 2, \ldots, n. \end{aligned}$$

Hence $\frac{\partial x^i}{\partial x^j} = \delta^i_j$.

2) $\delta^i_j \delta^j_k = \delta^i_k$. For,

$$\begin{aligned} \delta^i_j \delta^j_k &= \delta^i_1 \delta^1_k + \delta^i_2 \delta^2_k + \delta^i_3 \delta^3_k + \ldots + \delta^i_i \delta^i_k + \ldots + \delta^i_n \delta^n_k \\ &= 0 \times \delta^1_k + 0 \times \delta^2_k + \ldots + 1 \times \delta^i_k + \ldots + 0 \times \delta^n_k \\ &= \delta^i_k. \end{aligned}$$

3) $\delta^i_i = \delta^1_1 + \delta^2_2 + \ldots + \delta^n_n = 1 + 1 + \ldots + 1 = n.$

4) $\delta^i_j a^{jk} = a^{ik}$. For,

$$\begin{aligned} \delta^1_j a^{jk} &= \delta^1_1 a^{1k} + \delta^1_2 a^{2k} + \ldots + \delta^1_n a^{nk} \\ &= a^{1k} + 0 + \ldots + 0 \\ &= a^{1k}. \end{aligned}$$

Similarly, $\delta^2_j a^{jk} = a^{2k}, \delta^3_j a^{jk} = a^{3k}$ and hence $\delta^i_j a^{jk} = a^{ik}$.

5) $\delta^i_j a_{ik} = a_{jk}$.

6) $\delta^i_k a_{ji} = a_{jk}$.

7) $\delta^i_j a^j = a^i$.

8) $\delta_j^i a_i = a_j$.

9) $\delta_j^i a^{kj} = a^{ki}$.

We note that the Kronecker delta δ_j^i is called substitution operator because it allows us to replace one index by another as shown from properties (4) to (9).

I.4 Some Results of Determinants

It is assumed that the reader is familiar with the elementary properties of determinants. If the numbers i, j can take all positive integral values from 1 to n , the n^2 quantities a_j^i may be taken as elements of a determinant of order n , *viz,*

$$a = \begin{vmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \dots & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix} \tag{I.4.1}$$

which is a homogeneous polynomial of the nth degree in the elements. The index i denotes the row and j indicates column of the determinant $|a_j^i|$. If $a_j^i = a_i^j$ for all values of i and j, the determinant is said to be symmetric; while if $a_j^i = -a_i^j$ it is called skew-symmetric. If A_i^j denote the cofactor of the element a_j^i in the determinant a , then it is well known that the sum of the products of the elements of the i -th row (or column) is equal to a if $i = j$, and to zero if $i \neq j$. Hence

$$a_1^i A_j^1 + a_2^i A_j^2 + \ldots + a_n^i A_j^n = a\delta_j^i.$$

By summation convention we may write

$$a_k^i A_j^k = a\delta_j^i$$

and

$$a_i^k A_k^j = a\delta_i^j.$$

The determinant of the n^2 cofactors A_i^j of the elements of a is called the adjoint of a and is denoted by A *i.e.,* $A = \left|A_i^j\right|$. It is well known that $A = a^{n-1}$.

We shall now establish the following results on matrices and determinants of systems of second order. They will often be used later on.

(1) If $a_j^i b_p^j = c_p^i$, then $(a_j^i)(b_p^j) = (c_p^i)$ and $|a_j^i||b_p^j| = |c_p^i|$.

(2) If $a_k^i b_j^p c_p^k = d_j^i$, then $(a_k^i)(c_p^k)(b_j^p) = (d_j^i)$ and $|a_k^i||c_p^k||b_j^p| = |d_j^i|$.

We shall prove the results by taking the range of the indices from 1 to 2. But the results hold, in general, when the range is from 1 to n.

(1) We have $a_j^i b_p^j = a_1^i b_p^1 + a_2^i b_p^2$.

Hence, $c_p^i = a_1^i b_p^1 + a_2^i b_p^2$.

Therefore,

$$\begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix} = \begin{pmatrix} a_1^1 b_1^1 + a_2^1 b_1^2 & a_1^1 b_2^1 + a_2^1 b_2^2 \\ a_1^2 b_1^1 + a_2^2 b_1^2 & a_1^2 b_2^1 + a_2^2 b_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix}$$

Hence, $(c_p^i) = (a_j^i)(b_p^j)$.
Taking determinants of both sides, we get $|c_p^i| = |a_j^i||b_p^j|$, since $|AB| = |A||B|$, where A and B are two square matrices.
This proves (1). Similarly, we can prove (2).

I.5 Differentiation of a Determinant

We consider the determinant discussed in (I.4.1). Let the elements a_j^i be functions of the independent variables $x, y, z, \ldots$ etc. Then the derivative of a with respect to x is given by

$$\frac{\partial a}{\partial x} = \begin{vmatrix} \frac{\partial a_1^1}{\partial x} & \frac{\partial a_2^1}{\partial x} & \ldots & \frac{\partial a_n^1}{\partial x} \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1^n & a_2^n & \ldots & a_n^n \end{vmatrix} + \begin{vmatrix} a_1^1 & a_2^1 & \ldots & a_n^1 \\ \frac{\partial a_1^2}{\partial x} & \frac{\partial a_2^2}{\partial x} & \ldots & \frac{\partial a_n^2}{\partial x} \\ \ldots & \ldots & \ldots & \ldots \\ a_1^n & a_2^n & \ldots & a_n^n \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} a_1^1 & a_2^1 & \ldots & a_n^1 \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial a_1^n}{\partial x} & \frac{\partial a_2^n}{\partial x} & \ldots & \frac{\partial a_n^n}{\partial x} \end{vmatrix}. \qquad \text{(I.5.1)}$$

The first determinant in the right hand side of (I.5.1)

$$= \frac{\partial a_1^1}{\partial x} A_1^1 + \frac{\partial a_2^1}{\partial x} A_1^2 + \ldots + \frac{\partial a_n^1}{\partial x} A_1^n$$
$$= \frac{\partial a_i^1}{\partial x} A_1^i,$$

where A_j^i is the cofactor of a_i^j in the determinant $|a_j^i|$. The second term of (I.5.1)

$$= \frac{\partial a_1^2}{\partial x} A_2^1 + \frac{\partial a_2^2}{\partial x} A_2^2 + \ldots + \frac{\partial a_n^2}{\partial x} A_2^n$$
$$= \frac{\partial a_i^2}{\partial x} A_2^i.$$

Similarly the last determinant of right hand side of (I.5.1) $= \frac{\partial a_i^n}{\partial x} A_n^i$.
Thus

$$\frac{\partial a}{\partial x} = \frac{\partial a_i^1}{\partial x} A_1^i + \frac{\partial a_i^2}{\partial x} A_2^i + \ldots + \frac{\partial a_i^n}{\partial x} A_n^i$$
$$= A_j^i \frac{\partial a_i^j}{\partial x}.$$

Similarly, $\frac{\partial a}{\partial y} = A_j^i \frac{\partial a_i^j}{\partial y}$ and $\frac{\partial a}{\partial z} = A_j^i \frac{\partial a_i^j}{\partial z}$ and so on.

I.6 Linear Equations, Cramer's Rule

Let us consider a system of n linear equations

$$a_j^i x^j = b^i, \quad i, j = 1, 2, \ldots, n$$

in n-unknowns $x^1, x^2, \ldots, x^n$, where we have used summation convention. Let the coefficient determinant $|a_j^i| = a$ and the cofactor of a_j^i be denoted by A_i^j. Multiplying by A_i^k and summing for integral values of i from 1 to n we have

$$\begin{aligned} & a_j^i x^j A_i^k = b^i A_i^k \\ i.e., \quad & a\delta_j^k x^j = b^i A_i^k \\ i.e., \quad & a x^k = b^i A_i^k \\ i.e., \quad & x^k = \frac{b^i A_i^k}{a}, \qquad \text{provided that } a \neq 0. \end{aligned}$$

This is known as Cramer's rule for solving a system of linear equations and the solutions of the system exists provided that $a = |a_j^i| \neq 0$, *i.e.*, the linear transformation is non-singular.

I.7 Examples

Example 1. Find the number of components of a system of fourth order in which the indices takes values from 1 to 8.

The number of components of a system of i-th order in which the indices takes values from 1 to n is n^i. Here $n = 8$ and $i = 4$. Hence the required number is 8^4.

Example 2. Determine the number of types of a system of the 5th order.

A system of 5th order contains 5 indices. They may occur in the following ways :

1) All of them are upper,
2) all of them are lower,
3) one of them is upper and the remaining four are lower,
4) two of them are upper and the remaining three are lower,
5) three of them are upper and the remaining two are lower,
6) four of them are upper and the remaining one is lower.

Thus they are of 6 types, *viz.*,

$$a_{ijklm},\ a^{ijklm},\ a^i_{jklm},\ a^{ij}_{klm},\ a^{ijk}_{lm},\ a^{ijkl}_{m}.$$

Example 3. If $x^i = a^i_p y^p$ and $y^i = b^i_q z^q$, show that

$$x^i = a^i_q b^q_p z^p.$$

We have

$$x^i = a^i_p y^p, \tag{I.7.1}$$

$$y^i = b^i_q z^q. \tag{I.7.2}$$

By virtue of (I.7.2) we obtain from (I.7.1),

$$x^i = a^i_p b^p_q z^q.$$

Interchanging the dummy indices p and q we obtain the required result.

Example 4. If $a_{pqr}x^p x^q x^r = 0$ for all values of the independent variables $x^1, x^2, \ldots, x^n$ and a_{pqr}'s are constants show that

$$a_{kji} + a_{jki} + a_{ikj} + a_{ijk} + a_{kij} + a_{jik} = 0.$$

Since $x^1, x^2, \ldots, x^n$ are independent variables and a_{pqr}'s are constants, we obtain from $a_{pqr}x^p x^q x^r$ by differentiating with respect to x^i,

$$a_{pqr}\frac{\partial x^p}{\partial x^i}x^q x^r + a_{pqr}x^p\frac{\partial x^q}{\partial x^i}x^r + a_{pqr}x^p x^q\frac{\partial x^r}{\partial x^i} = 0,$$

or, $$a_{pqr}\delta^p_i x^q x^r + a_{pqr}x^p\delta^q_i x^r + a_{pqr}x^p x^q\delta^r_i = 0,$$

or, $$a_{iqr}x^q x^r + a_{pir}x^p x^r + a_{pqi}x^p x^q = 0.$$

Differentiating with respect to x^j we get

$$a_{iqr}\frac{\partial x^q}{\partial x^j}x^r + a_{iqr}x^q\frac{\partial x^r}{\partial x^j} + a_{pir}\frac{\partial x^p}{\partial x^j}x^r + a_{pir}x^p\frac{\partial x^r}{\partial x^j} + a_{pqi}\frac{\partial x^p}{\partial x^j}x^q + a_{pqi}x^p\frac{\partial x^q}{\partial x^j} = 0$$

or, $$a_{iqr}\delta^q_j x^r + a_{iqr}x^q\delta^r_j + a_{pir}\delta^p_j x^r + a_{pir}x^p\delta^r_j + a_{pqi}\delta^p_j x^q + a_{pqi}x^p\delta^q_j = 0$$

i.e., $$a_{ijr}x^r + a_{iqj}x^q + a_{jir}x^r + a_{pij}x^p + a_{jqi}x^q + a_{pji}x^p = 0.$$

Again differentiating with respect to x^k we obtain in a similar manner

$$a_{ijk} + a_{ikj} + a_{jik} + a_{kij} + a_{jki} + a_{kji} = 0.$$

Example 5. If a^r_s is a double system such that $a^r_m a^m_s = \delta^r_s$, show that $|a^r_s| = \pm 1$.

From $a^r_m a^m_s = \delta^r_s$, it follows that $|a^r_m a^m_s| = |\delta^r_s|$

i.e.,
$$|a^r_m|\,|a^m_s| = \begin{vmatrix} \delta^1_1 & \delta^1_2 & \ldots & \delta^1_n \\ \delta^2_1 & \delta^2_2 & \ldots & \delta^2_n \\ \vdots & \vdots & \ldots & \vdots \\ \delta^n_1 & \delta^n_2 & \ldots & \delta^n_n \end{vmatrix}.$$

i.e.,

$$\begin{vmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \dots & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix} \begin{vmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \dots & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix} = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix}$$

i.e.,

$$\begin{vmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \dots & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix}^2 = 1$$

i.e.,

$$\begin{vmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \dots & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix} = \pm 1.$$

Thus it follows that $|a_s^r| = \pm 1$.

Example 6. If y^i are n independent functions of the variables x^i, and z^i are n independent functions of y^i and if

$$u^i = v^j \frac{\partial x^i}{\partial y^j}, \quad v^i = w^j \frac{\partial y^i}{\partial z^j} \quad (i, j = 1, 2, \dots, n)$$

then show that $u^i = w^j \frac{\partial x^i}{\partial z^j}$.

We have
$$u^i = v^j \frac{\partial x^i}{\partial y^j} \tag{I.7.3}$$

and
$$v^i = w^j \frac{\partial y^i}{\partial z^j}. \tag{I.7.4}$$

From (I.7.3) and (I.7.4) we get

$$\begin{aligned} u^i &= v^j \frac{\partial x^i}{\partial y^j} \\ &= v^k \frac{\partial x^i}{\partial y^k} \qquad \text{(changing the dummy index } j \text{ by } k\text{)} \\ &= w^j \frac{\partial y^k}{\partial z^j} \frac{\partial x^i}{\partial y^k} \\ &= w^j \frac{\partial x^i}{\partial z^j}. \end{aligned}$$

Example 7. If the n^3 quantities B^i_{jk} are connected with the n^3 quantities A^h_{ml} by the equations

$$B^i_{jk} = A^h_{ml}\frac{\partial y^i}{\partial x^h}\frac{\partial x^m}{\partial y^j}\frac{\partial x^l}{\partial y^k}$$

then show that

$$B^i_{ik} = A^i_{il}\frac{\partial x^l}{\partial y^k}.$$

Putting $j = i$ in the given equations we have

$$\begin{aligned} B^i_{ik} &= A^h_{ml}\frac{\partial y^i}{\partial x^h}\frac{\partial x^m}{\partial y^i}\frac{\partial x^l}{\partial y^k} \\ &= A^h_{ml}\frac{\partial x^m}{\partial x^h}\frac{\partial x^l}{\partial y^k} \\ &= A^h_{ml}\delta^m_h\frac{\partial x^l}{\partial y^k} \\ &= A^m_{ml}\frac{\partial x^l}{\partial y^k} \\ &= A^i_{il}\frac{\partial x^l}{\partial y^k} \quad \text{(replacing the dummy index } m \text{ by } i\text{).} \end{aligned}$$

I.8 Exercises

Exercise 1 . If $A = \lambda_i x^i$ for all values of the independent variables $x^1, x^2, \ldots, x^n$ and λ_i's are constants, show that

$$\frac{\partial(\lambda_i x^i)}{\partial x^j} = \lambda_j.$$

Exercise 2 . If a^i_j is a double system satisfying the relation $a^i_j a^j_k = \delta^i_k$, show that either

$$|a^i_j - \delta^i_j| = 0$$

or,

$$|a^i_j + \delta^i_j| = 0.$$

Exercise 3 . If $a_{ij}x^i x^j = 0$ for all values of the independent variables $x^1, x^2, \ldots, x^n$ and a_{ij}'s are constants, show that

$$a_{pq} + a_{qp} = 0.$$

Exercise 4 . If $a = |a_j^i|$ and b_i^j is a^{-1} times the cofactor of a_j^i in the determinant $|a_j^i|$, show that

$$b_i^j a_k^i = \delta_k^j .$$

Exercise 5 . If i, j, k take all the values $1, 2, \ldots, n$, show that $\delta_j^i \delta_k^j = \delta_k^i, \delta_j^i \delta_i^j = n$. Also if A^i are n quantities and A^{ij} are n^2 quantities, show that $\delta_j^i A^j = A^i$ and $\delta_j^i A^{jk} = A^{ik}$.

Exercise 6 . If

$$|a_j^i| = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix}$$

and

$$|b_j^i| = \begin{vmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 & b_2^3 & b_3^3 \end{vmatrix},$$

then show that

$$|a_j^i| \, |b_m^k| = |c_j^i| ,$$

where

$$c_m^i = a_p^i b_m^p .$$

II Tensor Algebra

II.0 Introduction

This chapter presents the study of a certain type of space called an n-dimensional space which is suitable to develop the idea of tensors and algebra of tensors with respect to a certain transformation of coordinates.

II.1 n-dimensional Space

An ordered set of n independent real numbers $x^1, x^2, \ldots, x^n$ is called an n-tuple of real numbers and is denoted by $(x^1, x^2, \ldots, x^n)$. The set of all n-tuples of real numbers is said to be an n-dimensional arithmetic continuum and each n-tuple is called a point of this continuum. Such a continuum will be denoted by S_n. Sometimes an S_n is called an n-dimensional space because it can be endowed with the structure of an n-dimensional linear space. In n-dimensional real number space R^n we can have a number of extra structures such as metric structure, vector space structure etc. But for our purpose, we shall consider only the space S_n without any additional structure.

Let us introduce a suitable coordinate system in S_n. Thus if we take $(x^1, x^2, \ldots, x^n)$ are the coordinates of a point P of S_n in a coordinate system then there exists an one-to-one correspondence between the coordinates of points of S_n and the set of all coordinates like $(x^1, x^2, \ldots, x^n)$.

If $x^1, x^2, \ldots, x^n$ are the coordinates of a point P, we shall write that x^i are the coordinates of P and the corresponding coordinate system shall be denoted by (x^i).

A curve in S_n is defined as the collection of points satisfying the n equations

$$x^i = x^i(u), \quad i = 1, 2, \ldots, n,$$

u being a parameter and $x^i(u)$ denotes a function of u.

A surface in S_n is defined as the collection of points satisfying n equations

$$x^i = x^i(u, v), \quad i = 1, 2, \ldots, n$$

u, v being two independent parameters.

The totality of points whose coordinates are expressible as functions of m independent parameters is called a subspace of S_n of dimension m and may be

denoted by S_m. Any such subspace is said to be immersed in S_n. If $m = n-1$, S_m is called a hypersurface of S_n. An equation of the form

$$\phi(x^1, x^2, \ldots, x^n) = 0$$

determines a hypersurface; for such a relation reduces the number of independent variables to $n-1$. If c is an arbitrary constant, then

$$\phi(x^1, x^2, \ldots, x^n) = c$$

represents a family of hypersurfaces, each value of c determines a hypersurface. If the function ϕ is single-valued, one hypersurface of the family passes through each point of S_n.

If P be a point of S_n with coordinates x^i, $i = 1, 2, \ldots, n$ and Q be a neighbouring point with coordinates $x^i + dx^i$, $i = 1, 2, \ldots, n$ in a certain coordinate system (x^i) then the differentials dx^i are called the components of an infinitesimal displacement PQ in the coordinate system (x^i). Such a displacement determines a direction in S_n at P. The directions of the infinitesimal displacements dx^i and δx^i at P are regarded as being the same if the two sets of differentials are in proportion; *i.e.,* if

$$\frac{dx^1}{\delta x^1} = \frac{dx^2}{\delta x^2} = \ldots = \frac{dx^n}{\delta x^n}.$$

II.2 Transformation of Coordinates in S_n

Let P be any point of S_n with coordinates x^i, $i = 1, 2, \ldots, n$ in (x^i)-coordinate system and $\bar{x}^i$ be the coordinates of the same point with respect to another coordinate system $(\bar{x}^i)$ and the two systems be related by the equations

$$\bar{x}^i = \phi^i(x^1, x^2, \ldots, x^n), \quad i = 1, 2, \ldots, n \tag{II.2.1}$$

where each ϕ^i is a single-valued continuous function of the independent variables $x^1, x^2, \ldots, x^n$ and have continuous partial derivatives upto the required order and also the functional determinant

$$\begin{vmatrix} \frac{\partial \phi^1}{\partial x^1} & \frac{\partial \phi^1}{\partial x^2} & \ldots & \frac{\partial \phi^1}{\partial x^n} \\ \frac{\partial \phi^2}{\partial x^1} & \frac{\partial \phi^2}{\partial x^2} & \ldots & \frac{\partial \phi^2}{\partial x^n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial \phi^n}{\partial x^1} & \frac{\partial \phi^n}{\partial x^2} & \ldots & \frac{\partial \phi^n}{\partial x^n} \end{vmatrix} \neq 0, \tag{II.2.2}$$

which is called the Jacobian of the transformation (II.2.1) and is denoted by

$$J = \left|\frac{\partial \phi^i}{\partial x^i}\right| \text{ or } \left|\frac{\partial \bar{x}^i}{\partial x^i}\right| \text{ or } \left|\frac{\partial \bar{x}}{\partial x}\right|.$$

Since $J = \left|\frac{\partial \phi^i}{\partial x^i}\right| \neq 0$, the functions ϕ^i are independent and hence the equations (II.2.1) can be solved for the x^i as functions of $\bar{x}^i$ which gives

$$x^i = \psi^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n), \quad i = 1, 2, \ldots, n. \tag{II.2.3}$$

The relations (II.2.1) and (II.2.3) are called the transformations of coordinates of S_n. If we take (II.2.1) as the transformation of coordinates of a point P with coordinates x^i in (x^i)-coordinate system, then (II.2.1) determines the coordinates of the point P with coordinates $\bar{x}^i$ in the coordinate system $(\bar{x}^i)$. Then (II.2.3) may be taken as the inverse transformation of coordinates subject to the condition (II.2.2).

The nature of the Jacobian of the transformations (II.2.1) and (II.2.3) is given by the following :

Theorem 2.1 : If J and J' be the Jacobians of the transformations (II.2.1) and (II.2.3), then $JJ' = 1$.

Proof : Since J and J' are the Jacobians of the transformation (II.2.1) and (II.2.3) respectively we have

$$J = \left|\frac{\partial \bar{x}^i}{\partial x^j}\right| \text{ and } J' = \left|\frac{\partial x^i}{\partial \bar{x}^j}\right|.$$

Since $\bar{x}^i$'s are independent and x^i's are also independent, using formula of partial differentiation and summation convention we may write

$$\frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^j}, \quad k = 1, 2, \ldots, n$$

$$\text{or,} \quad \delta^i_j = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^j}$$

$$\text{i.e.,} \quad |\delta^i_j| = \left|\frac{\partial \bar{x}^i}{\partial x^k}\right|\left|\frac{\partial x^k}{\partial \bar{x}^j}\right|$$

$$\text{or,} \quad 1 = JJ'.$$

II.3 Invariants

Let $A(x^1, x^2, \ldots, x^n)$ be a function of n-coordinates x^i in a coordinate system (x^i) in S_n and $\bar{A}(x^1, x^2, \ldots, x^n)$ be its transform in another coordinate system $(\bar{x}^i)$. Then A is called an invariant or scalar of S_n with respect to the transformation (II.2.1) if

$$\bar{A}(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) = A(x^1, x^2, \ldots, x^n).$$

The partial derivatives of the invariant A with respect to the coordinate system (x^i) are the n functions given by

$$A_i = \frac{\partial A}{\partial x^i} \qquad \text{(II.3.1)}$$

and also its partial derivatives with respect to the coordinate system $(\bar{x}^i)$ are n functions $\bar{A}_i$ given by

$$\bar{A}_i = \frac{\partial \bar{A}}{\partial \bar{x}^i} = \frac{\partial A}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^i} \qquad \text{(since } \bar{A} = A\text{)}$$

and hence by virtue of (II.3.1) we get

$$\bar{A}_i = A_k \frac{\partial x^k}{\partial \bar{x}^i}. \tag{II.3.2}$$

The vector A_i in (II.3.1) is called the gradient of the scalar A and is denoted by grad$A$ and its components in the $(\bar{x}^i)$-coordinate system is given by (II.3.2).

The relations (II.3.1) and (II.3.2) leads us to define certain type of vectors, called covariant vectors in S_n.

II.4 Vectors

II.4.1 Covariant Vectors

A set of n functions A_i of n coordinates in a coordinate system (x^i) are said to form the components of a covariant vector if they transform to another coordinate system $(\bar{x}^i)$ according to the following rule :

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j. \tag{II.4.1}$$

It is obvious that the gradient of a scalar A, defined in II.3, is an example of a covariant vector.

From (II.4.1) we can obtain the expression of A_i in a coordinate system (x^i) in terms of the coordinate system $(\bar{x}^i)$ as follows :

Multiplying both sides of (II.4.1) by $\frac{\partial \bar{x}^i}{\partial x^k}$ we obtain

$$\begin{aligned} \frac{\partial \bar{x}^i}{\partial x^k} \bar{A}_i &= \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^j}{\partial \bar{x}^i} A_j \\ &= \delta^j_k A_j \qquad \left(\text{since, } \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^j}{\partial \bar{x}^i} = \delta^j_k\right) \\ &= A_k. \end{aligned}$$

Thus,

$$A_k = \frac{\partial \bar{x}^i}{\partial x^k} \bar{A}_i. \tag{II.4.2}$$

II.4.2 Contravariant Vectors

A set of n functions A^i of the n coordinates x^i in a coordinate system (x^i) are said to form the components of a contravariant vector if they transform to another coordinate system $(\bar{x}^i)$ according to the following rule :

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j. \tag{II.4.3}$$

Multiplying both sides of (II.4.3) by $\frac{\partial x^k}{\partial \bar{x}^i}$ and summing over the index i from 1 to n, we obtain

$$\frac{\partial x^k}{\partial \bar{x}^i}\bar{A}^i = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^j}A^j = \frac{\partial x^k}{\partial x^j}A^j = \delta_j^k A^j = A^k.$$

Hence

$$A^k = \frac{\partial x^k}{\partial \bar{x}^i}\bar{A}^i. \tag{II.4.4}$$

Now considering differentials of both the sides of (II.2.1) we get

$$d\bar{x}^i = \sum_{k=1}^{n}\frac{\partial \phi^i}{\partial x^k}dx^k = \sum_{k=1}^{n}\frac{\partial \bar{x}^i}{\partial x^k}dx^k,$$

and using summation convention we have

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^k}dx^k. \tag{II.4.5}$$

From (II.4.5), it follows that the differentials dx^i form the components of a contravariant vector in (x^i)-coordinate system whose components in the $(\bar{x}^i)$-coordinate system are $d\bar{x}^i$ and is given by (II.4.5). It follows immediately that $\frac{dx^i}{du}$ is also a contravariant vector, called the tangent vector to the curve $x^i = x^i(u)$.

Let us now consider a change of coordinates $x'^i = g^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$. Then the new components A'^i are given by

$$A'^i = \frac{\partial x'^i}{\partial \bar{x}^j}\bar{A}^j = \frac{\partial x'^i}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^k}A^k = \frac{\partial x'^i}{\partial x^k}A^k$$

which has the same form as (II.4.3). This shows that the transformations of contravariant vectors form a group.

The coordinates x^i will only behave like the components of a contravariant vector with respect to linear transformations of the type $\bar{x}^i = a_j^i x^j$, where a_j^i is a set of n^2 constants, which do not necessarily form the components of a mixed tensor. For in this case $\frac{\partial \bar{x}^i}{\partial x^j} = a_j^i$ and hence the transformation can be rewritten as $\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j}x^j$. But with respect to the general transformations of coordinates, x^i do not form the components of a contravariant vector. This means that if we select $A^i = x^i$, then the new components $\bar{A}^i$ with respect to the coordinate system $\bar{x}^i$ do not satisfy the equations $\bar{A}^i = \bar{x}^i$.

Example 1. There is no distinction between contravariant and covariant vectors when we restrict ourselves to orthogonal transformations of rectangular Cartesian system of coordinates.

In E_3 we consider orthogonal transformation of coordinates given by

$$\left.\begin{aligned} \bar{x}^1 &= l_1x^1 + m_1x^2 + n_1x^3 + a^1, \\ \bar{x}^2 &= l_2x^1 + m_2x^2 + n_2x^3 + a^2, \\ \bar{x}^3 &= l_3x^1 + m_3x^2 + n_3x^3 + a^3. \end{aligned}\right\} \tag{II.4.6}$$

Since this transformation is orthogonal, the corresponding matrix

$$A = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix}$$

is orthogonal. Hence $A^{-1} = A^T$. Therefore the inverse transformation can be written as

$$\left.\begin{aligned} x^1 &= l_1\bar{x}^1 + l_2\bar{x}^2 + l_3\bar{x}^3 + a_1, \\ x^2 &= m_1\bar{x}^1 + m_2\bar{x}^2 + m_3\bar{x}^3 + a_2, \\ x^3 &= n_1\bar{x}^1 + n_2\bar{x}^2 + n_3\bar{x}^3 + a_3. \end{aligned}\right\} \tag{II.4.7}$$

Now let us consider the contravariant transformation

$$\bar{A}^i = A^r \frac{\partial \bar{x}^i}{\partial x^r}, \quad r = 1,2,3$$

$$\text{or,} \quad \bar{A}^i = A^1 \frac{\partial \bar{x}^i}{\partial x^1} + A^2 \frac{\partial \bar{x}^i}{\partial x^2} + A^3 \frac{\partial \bar{x}^i}{\partial x^3}. \tag{II.4.8}$$

$$\begin{aligned} \text{That is,} \quad \bar{A}^1 &= A^1 \frac{\partial \bar{x}^1}{\partial x^1} + A^2 \frac{\partial \bar{x}^1}{\partial x^2} + A^3 \frac{\partial \bar{x}^1}{\partial x^3}, \\ \bar{A}^2 &= A^1 \frac{\partial \bar{x}^2}{\partial x^1} + A^2 \frac{\partial \bar{x}^2}{\partial x^2} + A^3 \frac{\partial \bar{x}^2}{\partial x^3}, \\ \bar{A}^3 &= A^1 \frac{\partial \bar{x}^3}{\partial x^1} + A^2 \frac{\partial \bar{x}^3}{\partial x^2} + A^3 \frac{\partial \bar{x}^3}{\partial x^3}. \end{aligned}$$

$$\left.\begin{aligned} \text{or,} \quad \bar{A}^1 &= A^1 l_1 + A^2 m_1 + A^3 n_1, \\ \bar{A}^2 &= A^1 l_2 + A^2 m_2 + A^3 n_2, \\ \bar{A}^3 &= A^1 l_3 + A^2 m_3 + A^3 n_3 \end{aligned}\right\}, \tag{II.4.9}$$

where we have used (II.4.6). Again let us consider the covariant transformation

$$\begin{aligned} \bar{A}_i &= A_r \frac{\partial x^r}{\partial \bar{x}^i} \\ &= A_1 \frac{\partial x^1}{\partial \bar{x}^i} + A_2 \frac{\partial x^2}{\partial \bar{x}^i} + A_3 \frac{\partial x^3}{\partial \bar{x}^i}. \end{aligned} \tag{II.4.10}$$

Hence we get in a similar manner,

$$\left.\begin{aligned} \bar{A}_1 &= A_1 l_1 + A_2 m_1 + A_3 n_1, \\ \bar{A}_2 &= A_1 l_2 + A_2 m_2 + A_3 n_2, \\ \bar{A}_3 &= A_1 l_3 + A_2 m_3 + A_3 n_3. \end{aligned}\right\} \tag{II.4.11}$$

Relations (II.4.9) and (II.4.11) define the same type of entity. Hence

$$\bar{A}^1 = \bar{A}_1, \quad \bar{A}^2 = \bar{A}_2, \quad \bar{A}^3 = \bar{A}_3 .$$

Hence there is no distinction between contravariant and covariant vectors.
The existence of an invariant is given by the following example.

Example 2. If λ_i and μ^i are the components of a covariant and contravariant vector respectively, then the sum $\lambda_i\mu^i$ is an invariant..

By the transformation law of covariant and contravariant vectors we have

$$\bar{\lambda}_i = \frac{\partial x^j}{\partial \bar{x}^i}\lambda_j$$

and

$$\bar{\mu}^i = \frac{\partial \bar{x}^i}{\partial x^k}\mu^k.$$

Hence

$$\begin{aligned}\bar{\lambda}_i\bar{\mu}^i &= \frac{\partial x^j}{\partial \bar{x}^i}\lambda_j\frac{\partial \bar{x}^i}{\partial x^k}\mu^k \\ &= \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^k}\lambda_j\mu^k \\ &= \delta^j_k\lambda_j\mu^k \\ &= \lambda_k\mu^k \\ &= \lambda_i\mu^i,\end{aligned}$$

changing the dummy index k by i. This means that $\lambda_i\mu^i$ is an invariant.

II.5 Tensors of Second Order

II.5.1 Contravariant Tensors of Order Two

A set A^{ij} of n^2 functions of n coordinates x^i in a coordinate system (x^i) are said to form the components of a contravariant tensor of order two (or of rank two) if they transform to another coordinate system $(\bar{x}^i)$ by the following rule :

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}A^{pq}. \tag{II.5.1}$$

We can obtain the components of the same tensor A^{ij} in $(\bar{x}^i)$-coordinate system in terms of the (x^i)-coordinate system as follows :

Multiplying both sides of (II.5.1) by $\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}$ we get

$$\begin{aligned}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\bar{A}^{ij} &= \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}A^{pq} \\ &= \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^q}A^{pq} \\ &= \delta^r_p\delta^s_q A^{pq} \\ &= A^{rs}.\end{aligned}$$

Hence

$$A^{rs} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\bar{A}^{ij}. \qquad \text{(II.5.2)}$$

The existence of a contravariant tensor of second order is guaranteed by the following example.

Example 3. If A^i and B^j are two contravariant vectors, then the n^2 quantities A^iB^j are the components of a contravariant tensor of order two.

Proof : Since A^i and B^j are two contravariant vectors we have by the transformation law,

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^k}A^k$$

and

$$\bar{B}^j = \frac{\partial \bar{x}^j}{\partial x^p}B^p.$$

Hence

$$\begin{aligned}\bar{A}^i\bar{B}^j &= \frac{\partial \bar{x}^i}{\partial x^k}A^k\frac{\partial \bar{x}^j}{\partial x^p}B^p \\ &= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^p}A^kB^p.\end{aligned}$$

If we write $A^kB^p = C^{kp}$ and $\bar{A}^i\bar{B}^j = \bar{C}^{ij}$, then we have from the above

$$\bar{C}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^p}C^{kp},$$

from which it follows by virtue of (II.5.1) that C^{ij} or A^iB^j are the components of a contravariant tensor of order 2.

Note : Similarly we can define a contravariant tensor of order p by considering a system of order p of type $A^{i_1 i_2 \ldots i_p}$.

II.5.2 Covariant Tensors of Order Two

A set A_{ij} of n^2 functions of n coordinates x^i in a coordinate system (x^i) are said to form the components of a covariant tensor of order 2 (or of rank 2) if they transform to another coordinate system $(\bar{x}^i)$ by the following rule :

$$\bar{A}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}A_{pq}. \qquad \text{(II.5.3)}$$

The components A_{ij} in the coordinate system (x^i) may be expressed in terms of another coordinate system $(\bar{x}^i)$ as follows :

Multiplying both sides of (II.5.3) by $\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}$ we get

$$\begin{aligned}\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\bar{A}_{ij} &= \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}A_{pq} \\ &= \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial \bar{x}^j}{\partial x^s}\frac{\partial x^q}{\partial \bar{x}^j}A_{pq}\end{aligned}$$

$$i.e., \quad \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\bar{A}_{ij} = \delta_r^p\delta_s^q A_{pq} = A_{rs}.$$

Thus,

$$A_{rs} = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\bar{A}_{ij}. \tag{II.5.4}$$

The existence of a covariant tensor of order 2 may be guaranteed by taking two covariant vectors A_i and B_j and proceeding in a similar manner as in the case of a contravariant tensor of order 2.

Note : A covariant tensor of order q can be defined similarly by considering a system of order q of type $A_{i_1 i_2 \ldots i_q}$.

II.5.3 Mixed Tensors of Order Two

A set A_j^i of n^2 functions of n coordinates x^i in a coordinate system (x^i) are said to form the components of a mixed tensor of order 2 (or of rank 2) if they transform into another coordinate system $(\bar{x}^i)$ by the following rule :

$$\bar{A}_j^i = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^j}A_q^p. \tag{II.5.5}$$

In a similar manner like (II.5.4) we can obtain the formula for expressing the components A_j^i in a coordinate system (x^i) in terms of another coordinate system $(\bar{x}^i)$ as follows :

$$A_j^i = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial \bar{x}^q}{\partial x^j}\bar{A}_q^p. \tag{II.5.6}$$

Now if we choose A_q^p to be the Kronecker delta δ_q^p in (II.5.5), then we get

$$\bar{A}_j^i = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^j}\delta_q^p = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^p}{\partial \bar{x}^j} = \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta_j^i.$$

This implies that the Kronecker delta δ_j^i is a mixed tensor of the second order whose components in any other system again form the Kronecker delta. This justifies the placing of one index as a subscript and the other as a superscript.

Note 1: The indices are placed on the tensors as superscripts when they denote contravariance and as subscripts when they denote covariance. In particular, a mixed tensor A_j^i transforms like a contravariant vector with respect to the index i and like a covariant vector with respect to the index j. Consequently, i is placed as a superscript and j is placed as a subscript.

Note 2: If we select n^2 quantities $\delta_{ij} = \delta_j^i$ as the components of a covariant

vector in a coordinate system (x^i), the components in $(\bar{x}^i)$ system are given by $\bar{\delta}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^k}{\partial \bar{x}^j}$. This implies that the transformed components $\bar{\delta}_{ij}$ do not form the Kronecker delta.

Note 3: We can obtain a mixed tensor of second order by taking a contravariant vector A^i and a covariant vector B_j.

II.6 Mixed Tensors of Type (p, q)

A set of n^{p+q} functions $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ of the n coordinates in a coordinate system (x^i) are said to form the components of a mixed tensor of the $(p+q)$-th order, contravariant of the p-th order and covariant of the q-th order, if they transform according to the following rule into another coordinate system $(\bar{x}^i)$ given by

$$\bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = \frac{\partial \bar{x}^{i_1}}{\partial x^{r_1}}\frac{\partial \bar{x}^{i_2}}{\partial x^{r_2}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{r_p}}\frac{\partial x^{s_1}}{\partial \bar{x}^{j_1}}\frac{\partial x^{s_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{s_q}}{\partial \bar{x}^{j_q}} A^{r_1 r_2 \ldots r_p}_{s_1 s_2 \ldots s_q}. \qquad \text{(II.6.1)}$$

This formula is nothing but merely a combination of (II.5.1) with respect to contravariant indices and (II.5.3) with respect to covariant indices. Such a tensor sometimes called a tensor of type (p, q). Thus a contravariant vector is a tensor of type $(1, 0)$ or of order 1 or of rank 1 and a covariant vector is a tensor of type $(0, 1)$ while an invariant or a scalar may be regarded as a tensor of type $(0, 0)$ or of order zero.

However, it may be noted that the components of a tensor change under transformation of coordinates, but the entity called tensors does not change under coordinate transformation.

Note : Tensor is the generalization of a vector.

II.7 Zero Tensor

A tensor whose components are all zero in a coordinate system is called a zero tensor. It can be easily seen that if the components are zero in a certain coordinate system then they will be zero in every other coordinate system. So we may define the zero tensor as a tensor whose components are all zero in every coordinate system.

II.8 Tensor Field

At each point of S_n a tensor has definite numerical values; so that there is one tensor corresponding to each point of S_n. The system of tensors, one at each point where the functions are defined, is called a tensor field. The distinction between a tensor and a tensor field is the same as that between a vector and a vector field. It is usual to refer to a tensor field simply as a tensor.

II.9 Algebra of Tensors

The addition and subtraction of two tensors of the same type is a tensor of the same type.

Theorem 9.1 : Let $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ and $B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ be the components of two tensors of type (p, q) of S_n. Then their sum

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} + B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$$

are the components of another tensor of type (p, q).

Proof : Let $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ and $B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ be the components of two tensors in a coordinate system (x^i) and $\bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ and $\bar{B}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ be their components in another coordinate system $(\bar{x}^i)$. Then

$$\left(\bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} + \bar{B}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}\right) = \frac{\partial \bar{x}^{i_1}}{\partial x^{r_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{r_2}} \cdots \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{r_p}} \frac{\partial x^{s_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{s_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{s_q}}{\partial \bar{x}^{j_q}} \left(A^{r_1 r_2 \ldots r_p}_{s_1 s_2 \ldots s_q} + B^{r_1 r_2 \ldots r_p}_{s_1 s_2 \ldots s_q}\right). \tag{II.9.1}$$

From (II.9.2) it follows that

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} + B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$$

are the components of a tensor of type (p, q). This proves the theorem.

Note 1 : The algebraic operation by which the sum of two tensors of the same type is obtained is called the addition of the tensors.

Note 2 : In a similar manner, we can show that the difference

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} - B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$$

are the components of a tensor of type (p, q) and the algebraic operation by which it is obtained is called the subtraction of the tensors.

Theorem 9.2 : If $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ are the components of a tensor of type (p, q) of S_n and ϕ be a scalar, then $\phi A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ are the components of another tensor of type (p, q).

Proof : Let $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ be the components of a tensor at a point P of S_n in a coordinate system (x^i) and $\bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ be the components of the same tensor in another coordinate system $(\bar{x}^i)$. Let ϕ be the scalar in (x^i) system and $\bar{\phi}$ be in $(\bar{x}^i)$-coordinate system. Then we have

$$\begin{aligned} \bar{\phi} \bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} &= \bar{\phi} \frac{\partial \bar{x}^{i_1}}{\partial x^{r_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{r_2}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{r_p}} \frac{\partial x^{s_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{s_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{s_q}}{\partial \bar{x}^{j_q}} A^{r_1 r_2 \ldots r_p}_{s_1 s_2 \ldots s_q} \\ &= \frac{\partial \bar{x}^{i_1}}{\partial x^{r_1}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{r_p}} \frac{\partial x^{s_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{s_q}}{\partial \bar{x}^{j_q}} \phi A^{r_1 r_2 \ldots r_p}_{s_1 s_2 \ldots s_q}, \qquad \text{(since } \bar{\phi} = \phi\text{).} \end{aligned} \tag{II.9.2}$$

From (II.9.2), it follows that $\phi A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ are the components of a tensor of type (p, q).

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Note : The algebraic operation done in Theorem (II.9.2) is called the multiplication of a tensor by a scalar.

Theorem 9.3 : If $a_{ij}A^iB^j = 0$ for two distinct arbitrary vectors A^i and B^j, then $a_{ij} = 0$.

Proof : The given relation can be written as

$$\begin{array}{rlll} & a_{11}A^1B^1 + a_{12}A^1B^2 & +\ldots+ & a_{1n}A^1B^n \\ + & a_{21}A^2B^1 + a_{22}A^2B^2 & +\ldots+ & a_{2n}A^2B^n \\ + & \ldots & & \\ + & a_{n1}A^nB^1 + a_{n2}A^nB^2 & +\ldots+ & a_{nn}A^nB^n = 0. \end{array} \qquad \text{(II.9.3)}$$

Among the above n^2 terms, in n terms the indices are equal, *i.e.*, $i = j$; in the rest $n^2 - n$ terms $i \neq j$, *i.e.*, the indices are different.

Since A^i and B^j are arbitrary we are free to choose them at will. First let $A^i = (1, 0, \ldots, 0)$, $B^i = (1, 0, \ldots, 0)$.
Then from (II.9.3) we get

$$\begin{array}{rlll} & a_{11}(1)(1) + a_{12}(1)(0) & +\ldots+ & a_{1n}(1)(0) \\ + & a_{21}(0)(1) + a_{22}(0)(0) & +\ldots+ & a_{2n}(0)(0) \\ + & \ldots & & \\ + & a_{n1}(0)(1) + a_{n2}(0)(0) & +\ldots+ & a_{nn}(0)(0) = 0 \end{array}$$

(since $A^1 = 1$, $A^2 = A^3 = \ldots = A^n = 0$, $B^1 = 1$, $B^2 = B^3 = \ldots = B^n = 0$).
This gives $a_{11} = 0$.

Hence (II.9.3) reduces to

$$\begin{array}{rlll} & a_{12}A^1B^2 & +\ldots+ & a_{1n}A^1B^n \\ + & a_{21}A^2B^1 + a_{22}A^2B^2 & +\ldots+ & a_{2n}A^2B^n \\ + & \ldots & & \\ + & a_{n1}A^nB^1 + a_{n2}A^nB^2 & +\ldots+ & a_{nn}A^nB^n = 0. \end{array} \qquad \text{(II.9.4)}$$

In the similar fashion choosing A^i and B^j as unit vectors, both with their r-th component $(1 \leq r \leq n)$ equal to 1 and others being zero, we get $a_{rr} = 0$.
Thus $a_{rr} = 0$ for all $r = 1, 2, \ldots, n$.

Next we take $A^i = (0, 1, 0, \ldots, 0)$ and $B^j = (1, 0, \ldots, 0)$.
And hence we can get

$$a_{21} = 0.$$

Thus taking A^i and B^j as unit vectors
$A^l = 1 = B^m$ $(1 \leq l \leq n,\ 1 \leq m \leq n,\ l \neq m)$
and all the other components of A^i and B^j are zero we get

$$a_{lm} = 0.$$

Thus $a_{ij} = 0$ for all $i, j = 1, 2, \ldots, n$.

II.10 Equality of Two Tensors

Two tensors are said to be equal in a certain coordinate system if they are of the same type and their corresponding components in the coordinate system are equal. Thus, if $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ and $B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ are the components of two equal tensors in the same coordinate system, then we must have

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}.$$

If two tensors are equal in a certain coordinate system, then they are equal in every coordinate system as shown by the following theorem.

Theorem 10.1 : If two tensors are equal in a coordinate system, then they are also equal in every other coordinate system.

Proof : Let $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ and $B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ be the components of two equal tensors in a coordinate system (x^i). Then

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}.$$

This implies that

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} - B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} \qquad \text{(II.10.1)}$$

are the components of a zero tensor in the coordinate system (x^i). Since the components of a zero tensor are equal to zero in every other coordinate system, it follows that the difference (II.10.1) must be a zero tensor in every other coordinate system. Hence the tensors are equal in every other coordinate system. This proves the theorem.

Thus we may say that two tensors of the same type are equal if all their corresponding components are equal in every coordinate system.

The order of indices in a tensor is important. The tensor A_{ij} is not necessarily the same as the tensor A_{ji}. In the next section we shall discuss about two important types of tensors.

II.11 Symmetric and Skew-symmetric Tensors

II.11.1 Symmetric Tensors

If in a coordinate system two contravariant or covariant indices of a tensor can be interchanged without altering the tensor, then it is said to be symmetric with respect to these two indices in this coordinate system.

For example, a tensor with components A_{ijk} is symmetric with respect to the indices j and k if and only if $A_{ijk} = A_{ikj}$.

The following theorem shows that the symmetric property remains unchanged under coordinate transformation.

Theorem 11.1: If a tensor is symmetric with respect to two contravariant or covariant indices in any coordinate system, then it remains unchanged with respect to these two indices in any other coordinate system.

Proof : Let us consider a covariant tensor of type $(0, q)$ with components $A_{i_1 i_2 \ldots i_q}$ in a coordinate system (x^i) and $\bar{A}_{i_1 i_2 \ldots i_q}$ in another coordinate system $(\bar{x}^i)$. We also suppose that the tensor is symmetric in the indices i_r and i_s in the (x^i)-coordinate system. Then we have

$$A_{i_1 i_2 \ldots i_{r-1} i_r i_{r+1} \ldots i_{s-1} i_s i_{s+1} \ldots i_q} = A_{i_1 i_2 \ldots i_{r-1} i_s i_{r+1} \ldots i_{s-1} i_r i_{s+1} \ldots i_q}. \qquad \text{(II.11.1)}$$

Now,

$$\begin{aligned}
\bar{A}_{i_1 i_2 \ldots i_r \ldots i_s \ldots i_q} &= \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} \cdots \frac{\partial x^{j_r}}{\partial \bar{x}^{i_r}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{i_s}} \cdots \frac{\partial x^{j_q}}{\partial \bar{x}^{i_q}} A_{j_1 j_2 \ldots j_r \ldots j_s \ldots j_q} \\
&= \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{i_s}} \cdots \frac{\partial x^{j_r}}{\partial \bar{x}^{i_r}} \cdots \frac{\partial x^{j_q}}{\partial \bar{x}^{i_q}} A_{j_1 j_2 \ldots j_s \ldots j_r \ldots j_q} \text{ (by II.11.1)} \\
&= \bar{A}_{i_1 i_2 \ldots i_s \ldots i_r \ldots i_q}.
\end{aligned}$$

This implies that symmetry with respect to i_r and i_s also holds in the $(\bar{x}^i)$ coordinate system. Similar results holds for a contravariant and a mixed tensor of any order. This proves the theorem.

Definition : If every pair of contravariant or covariant indices of a tensor be interchanged without altering the tensor, then it is said to be symmetric in every pair of such indices or simply a symmetric tensor. For example, a tensor A_{ijk} is a symmetric tensor if and only if

$$A_{ijk} = A_{jik} = A_{ikj} = A_{kji}.$$

A symmetric tensor of the second order has at most $\frac{1}{2}n(n+1)$ different components. However it may be remembered that symmetry cannot, in general, be defined for a tensor with respect to two indices of which one is contravariant and the other is covariant. But the tensor δ^i_j has the property that it is symmetric in i and j and this symmetry is preserved under transformation of coordinates.

II.11.2 Skew-symmetric Tensors

If in a coordinate system two contravariant or covariant indices of a tensor be interchanged and as a result each of its components is altered in sign but not in magnitude, then the tensor is said to be a skew-symmetric tensor with respect to these indices in this coordinate system. For example, the tensor A_{ij} is skew-symmetric with respect to i and j if $A_{ij} = -A_{ji}$.

It can be easily seen that the skew-symmetry with respect to two contravariant or covariant indices in a coordinate system remains unchanged in every other coordinate system with respect to those indices.

Definition : If every pair of contravariant or covariant indices of a tensor be interchanged and as a result each of its components is altered in sign but not in

magnitude, then the tensor is said to be skew-symmetric in every pair of such indices or simply a skew-symmetric tensor.

The skew-symmetry can not be defined for a tensor with respect to two indices of which one is contravariant and the other is covariant. A skew-symmetric tensor A^{ij} of the second order has at most $\frac{1}{2}n(n-1)$ different arithmetical components, as all the quantities A^{ii} are zero (no summation). The several components of a skew-symmetric tensor of the n-th order are either zero or differ merely in sign.

Theorem 11.2 : Any covariant tensor of second order can be expressed uniquely as the sum of a symmetric and a skew-symmetric tensor of the second order.

Proof : Let a_{ij} be the components of a covariant tensor of second order. Then we may write

$$\begin{aligned} a_{ij} &= \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji}) \\ &= A_{ij} + B_{ij}, \quad \text{say,} \end{aligned}$$

where

$$A_{ij} = \frac{1}{2}(a_{ij}+a_{ji})$$

and

$$B_{ij} = \frac{1}{2}(a_{ij}-a_{ji}).$$

Since a_{ij} is a covariant tensor of the second order, a_{ji} is also a tensor of the same order and hence by addition and subtraction of tensors, it follows that A_{ij} and B_{ij} are covariant tensors of the second order. Also, we have $A_{ij} = A_{ji}$ and $B_{ij} = -B_{ji}$ and hence A_{ij} is symmetric and B_{ij} is skew-symmetric.
Let us suppose that

$$a_{ij} = P_{ij} + Q_{ij} \tag{II.11.2}$$

where P_{ij} is symmetric and Q_{ij} is skew-symmetric. Then

$$a_{ji} = P_{ji} + Q_{ji} = P_{ij} - Q_{ij}, \tag{II.11.3}$$

since P_{ij} is symmetric and Q_{ij} is skew-symmetric. From (II.11.2) and (II.11.3) we get

$$P_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) = A_{ij}$$

and

$$Q_{ij} = \frac{1}{2}(a_{ij}-a_{ji}) = B_{ij}.$$

Hence, the decomposition is unique.
Note : This theorem is also true for a tensor of type $(2,0)$ *i.e.*, for a contravariant tensor of the second order.

II.12 Outer Multiplication and Contraction

II.12.1 Outer Multiplication

We shall now consider an algebraic operation, called outer multiplication of tensors (even of different types) which produces a tensor of higher rank than the rank of the given tensors.

Theorem 12.1 : If $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ and $B^{k_1 k_2 \ldots k_r}_{l_1 l_2 \ldots l_s}$ are the components of two tensors of type (p, q) and (r, s) respectively (r and s not being both zero), then their outer product is a tensor of type $(p+r, q+s)$ whose components are the quantities $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} B^{k_1 k_2 \ldots k_r}_{l_1 l_2 \ldots l_s}$.

Proof : According to the transformation law, we have

$$\bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = \frac{\partial \bar{x}^{i_1}}{\partial x^{t_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{t_2}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{t_p}} \frac{\partial x^{h_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{h_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{h_q}}{\partial \bar{x}^{j_q}} A^{t_1 t_2 \ldots t_p}_{h_1 h_2 \ldots h_q}, \tag{II.12.1}$$

$$\bar{B}^{k_1 k_2 \ldots k_r}_{l_1 l_2 \ldots l_s} = \frac{\partial \bar{x}^{k_1}}{\partial x^{m_1}} \frac{\partial \bar{x}^{k_2}}{\partial x^{m_2}} \cdots \frac{\partial \bar{x}^{k_r}}{\partial x^{m_r}} \frac{\partial x^{u_1}}{\partial \bar{x}^{l_1}} \frac{\partial x^{u_2}}{\partial \bar{x}^{l_2}} \cdots \frac{\partial x^{u_s}}{\partial \bar{x}^{l_s}} B^{m_1 m_2 \ldots m_r}_{u_1 u_2 \ldots u_s}. \tag{II.12.2}$$

From (II.12.1) and (II.12.2) we get

$$\bar{C}^{i_1 i_2 \ldots i_p k_1 k_2 \ldots k_r}_{j_1 j_2 \ldots j_q l_1 l_2 \ldots l_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{t_1}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{t_p}} \frac{\partial \bar{x}^{k_1}}{\partial x^{m_1}} \cdots \frac{\partial \bar{x}^{k_r}}{\partial x^{m_r}} \frac{\partial x^{h_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{h_q}}{\partial \bar{x}^{j_q}} \frac{\partial x^{u_1}}{\partial \bar{x}^{l_1}} \cdots \frac{\partial x^{u_s}}{\partial \bar{x}^{l_s}} C^{t_1 t_2 \ldots t_p m_1 m_2 \ldots m_r}_{h_1 h_2 \ldots h_q u_1 u_2 \ldots u_s} \tag{II.12.3}$$

where

$$\bar{C}^{i_1 i_2 \ldots i_p k_1 k_2 \ldots k_r}_{j_1 j_2 \ldots j_q l_1 l_2 \ldots l_s} = \bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} \bar{B}^{k_1 k_2 \ldots k_r}_{l_1 l_2 \ldots l_s}$$

and

$$C^{t_1 t_2 \ldots t_p m_1 m_2 \ldots m_r}_{h_1 h_2 \ldots h_q u_1 u_2 \ldots u_s} = A^{t_1 t_2 \ldots t_p}_{h_1 h_2 \ldots h_q} B^{m_1 m_2 \ldots m_r}_{u_1 u_2 \ldots u_s}.$$

From (II.12.3) it follows that $C^{t_1 t_2 \ldots t_p m_1 m_2 \ldots m_r}_{h_1 h_2 \ldots h_q u_1 u_2 \ldots u_s}$ are the components of a tensor of type $(p+r, q+s)$.

II.12.2 Contraction

We now consider an algebraic operation called contraction which can only be defined for mixed tensors by identifying two indices one of upper index and another lower index.

Theorem 12.2 : If A^{ij}_{klm} are the components of a mixed tensor of type $(2, 3)$, then the quantities A^{ij}_{kim} obtained by replacing the subscript l by the superscript i and taking summation over i, are the components of a tensor of type $(1, 2)$.

Proof : By the transformation law, we have

$$\bar{A}^{ij}_{klm} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l}\frac{\partial x^t}{\partial \bar{x}^m}A^{pq}_{rst}. \qquad \text{(II.12.4)}$$

Putting $l = i$ in (II.12.4) we get

$$\begin{aligned}\bar{A}^{ij}_{kim} &= \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial x^t}{\partial \bar{x}^m}A^{pq}_{rst} \\ &= \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^m}\delta^s_p A^{pq}_{rst},\end{aligned}$$

where $\delta^s_p = 1$ for $s = p$ and $\delta^s_p = 0$ for $s \neq p$.
Using summation convention we get

$$\bar{A}^{ij}_{kim} = \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^m}A^{pq}_{rpt}$$

i.e.,

$$\bar{B}^{j}_{km} = \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^t}{\partial \bar{x}^m}B^{q}_{rt} \qquad \text{(II.12.5)}$$

where $\bar{B}^j_{km} = \bar{A}^{ij}_{kim}$ and $B^q_{rt} = A^{pq}_{rpt}$.

From (II.12.5), it follows that B^j_{km} are the components of a tensor of type $(1,2)$. Thus we have started with a tensor of type $(2,3)$ and after using the operation, contraction, we get a tensor of type $(2-1, 3-1)$ *i.e.*, of type $(1,2)$. Here the tensor A^{ij}_{kim} is called a contracted tensor of the given tensor and the operation by which it is obtained is called contraction.

In general, if $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ are the components of a tensor of type (p,q), $p \neq 0, q \neq 0$, then by replacing any one upper index out of $i_1, i_2, \ldots, i_p$ and any lower index out of $j_1, j_2, \ldots, j_q$ by the same index and performing summation over that index, we obtain the components of a tensor of type $(p-1, q-1)$. This process may continue p times if $p \leq q$ and q times if $q \leq p$.

Note : Contraction of m pairs of indices of a mixed tensor of type (p,q) yields a tensor of type $(p-m, q-m)$ whose rank or order is less than that of the original tensor by $2m$. Thus contraction can reduce the order or rank of a tensor by an even number. Hence if we apply contraction to a mixed tensor of type $(1,1)$ then we obtain a tensor of type $(0,0)$ *i.e.*, an invariant.

II.13 Inner Multiplication

Inner multiplication is an operation which is a combination of two basic operations, outer multiplication and contraction. The resultant object is a tensor and is called the inner product of two tensors. Inner multiplication is also known as transvection.

If an outer product of two tensors be contracted with respect to an upper index of one factor and a lower index of the other, then a tensor is obtained which is called an inner product of the two tensors. Thus if we contract the outer product $A_{ij}B^{kl}$ for the indices i and k, then it produces the tensor $A_{ij}B^{il}$ which is of type

$(1,1)$. Again, if the outer product A_iB^j of the vectors A_i and B^j be contracted for the indices i and j, then it produces a tensor A_iB^i of type $(0,0)$ *i.e.*, an invariant or a scalar. This inner product is called the **scalar product of the vectors A_i and B^j** because it is a scalar. It is also noted that in S_n, the scalar product of two vectors is defined for two vectors of opposite variance.

II.14 Quotient Law of Tensors

We have seen that the product of two tensors is a tensor. Let us assume that the product of two quantities is a tensor and if one of them is a tensor, is the other quantity a tensor? There is a law, called quotient law which gives the answer to this question in the affirmative sense.

At first, we shall state and prove this law for tensors of order 1 to 2 and then give the same in its general formulation.

II.14.1 Quotient Law for Tensors of First Order and Type $(0,1)$

Theorem 14.1 : Let $B_i, i = 1,2,\ldots,n$ be n quantities in a certain coordinate system. If $A^i, i = 1,2,\ldots n$ are the components of an arbitrary tensor of type $(1,0)$ such that A^iB_i is an invariant, then B_i is a tensor of type $(0,1)$.

Proof : Since A^iB_i is an invariant, for any two systems of coordinates (x^i) and $(\bar{x}^i)$, we have

$$\bar{A}^i\bar{B}_i = A^iB_i. \tag{II.14.1}$$

Since A^i is a tensor of type $(1,0)$, by transformation law, we have

$$A^i = \bar{A}^p\frac{\partial x^i}{\partial \bar{x}^p}.$$

By virtue of this (II.14.1) can be written as

$$\bar{A}^i\bar{B}_i = \bar{A}^p\frac{\partial x^i}{\partial \bar{x}^p}B_i.$$

Changing the dummy indices p and i by i and p respectively, we get

$$\bar{A}^i\bar{B}_i = \bar{A}^i\frac{\partial x^p}{\partial \bar{x}^i}B_p$$

$$i.e.,\quad \bar{A}^i\left(\bar{B}_i - B_p\frac{\partial x^p}{\partial \bar{x}^i}\right) = 0.$$

Since $\bar{A}^i$ are the components of an arbitrary tensor, we have

$$\bar{B}_i - \frac{\partial x^p}{\partial \bar{x}^i}B_p = 0$$

$$i.e.,\quad \bar{B}_i = \frac{\partial x^p}{\partial \bar{x}^i}B_p. \tag{II.14.2}$$

From (II.14.2), it follows that B_i are the components of a covariant vector *i.e.*, B_i are the components of a tensor of type $(0,1)$. This proves the result.

II.14.2 Quotient Law for Tensors of First Order and Type $(1,0)$

Theorem 14.2 : Let $A^i, i = 1, 2, \ldots n$ be n quantities in a certain coordinate system. If $B_i, i = 1, 2, \ldots, n$ are the components of an arbitrary tensor of type $(0,1)$ such that $A^i B_i$ is an invariant, then A^i is a tensor of type $(1,0)$.

Proof : Proof is similar to the case (II.14.1).

II.14.3 Quotient Law for Covariant Tensors of Second Order or of Type $(0,2)$

Theorem 14.3 : Let a_{ij} $(i, j = 1, 2, \ldots n)$ be n^2 quantities in a certain coordinate system. If A^i and B^j are two arbitrary tensors of type $(1,0)$ such that $a_{ij} A^i B^j$ is an invariant, then a_{ij} are the components of a tensor of type $(0,2)$.

Proof : According to the invariant hypothesis, we have

$$\bar{a}_{ij} \bar{A}^i \bar{B}^j = a_{ij} A^i B^j. \tag{II.14.3}$$

Also we have

$$A^i = \bar{A}^p \frac{\partial x^i}{\partial \bar{x}^p}$$

$$\text{and} \quad B^j = \bar{B}^l \frac{\partial x^j}{\partial \bar{x}^l}.$$

Hence (II.14.3) can be written as,

$$\bar{a}_{ij} \bar{A}^i \bar{B}^j = a_{ij} \bar{A}^p \bar{B}^l \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^l}.$$

Changing the dummy indices p and l by i and j respectively, we get

$$\bar{a}_{ij} \bar{A}^i \bar{B}^j = a_{pl} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \bar{A}^i \bar{B}^j$$

$$i.e., \quad \left(\bar{a}_{ij} - a_{pl} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \right) \bar{A}^i \bar{B}^j = 0.$$

Since $\bar{A}^i$ and $\bar{B}^j$ are the components of two arbitrary tensors, we get

$$\bar{a}_{ij} - a_{pl} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} = 0$$

$$i.e., \quad \bar{a}_{ij} = a_{pl} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j}.$$

This shows that a_{ij} are the components of a tensor of type $(0,2)$. This completes the proof.

II.14.4 Quotient Law for Contravariant Tensors of Second Order or of Type $(2,0)$

Theorem 14.4 : Let a^{ij} $(i,j=1,2,\ldots n)$ be n^2 quantities in a certain reference frame. If A_i and B_j are two arbitrary tensors of type $(0,1)$ such that $a^{ij}A_iB_j$ is an invariant, then a^{ij} are the components of a tensor of type $(2,0)$.

II.14.5 Quotient Law in General Form

Theorem 14.5 : Let $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ (each of the indices taking values from 1 to n) be n^{p+q} quantities in a certain reference frame. If $A^1_{i_1}, A^2_{i_2}, \ldots, A^r_{i_r}; B^{j_1}_1, B^{j_2}_2, \ldots, B^{j_s}_s$ are the components of arbitrary covariant and contravariant vectors respectively such that

$$A^{i_1 i_2 \ldots i_r i_{r+1} \ldots i_p}_{j_1 j_2 \ldots j_s j_{s+1} \ldots j_q} A^1_{i_1} A^2_{i_2} \ldots A^r_{i_r} B^{j_1}_1 B^{j_2}_2 \ldots B^{j_s}_s \quad (r \le p, s \le q)$$

are components of a tensor of type $(p-r, q-s)$, then $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ will form the components of a tensor of type (p,q).

The above quotient laws give the general feature, but when order of tensor is increased the application of the quotient law in the above forms become cumbersome. So we need to give an alternative formulation of the quotient law, which may be convenient and which is sometimes called the quotient theorem.

Alternative Form of the Quotient Law

If the result of inner multiplication of some quantities by an arbitrary tensor of known type is a tensor, then the given quantities will be the components of a tensor (types can be easily found). For example, let B^r_s be an arbitrary tensor of type $(1,1)$ and $A(i,j,k)$ be n^3 given quantities in a certain reference frame. If $B^r_s A(s,j,k)$ (summation over s) is a tensor of type $(1,2)$, then $A(i,j,k)$ is a tensor of type $(1,2)$, *i.e.*, $A(i,j,k)$ is of type A^i_{jk}.

II.15 Reciprocal Tensor of a Tensor

Let a_{ij} be a tensor of type $(0,2)$ such that $|a_{ij}| \neq 0$. We denote by b^{ij} the cofactor of a_{ij} in $|a_{ij}|$ divided by $|a_{ij}|$, *i.e.*,

$$b^{ij} = \frac{\text{cofactor of } a_{ij} \text{ in } |a_{ij}|}{|a_{ij}|}. \qquad \text{(II.15.1)}$$

From the theory of determinants and in virtue of (II.15.1), we have

$$a_{ij} b^{kj} = \delta^k_i. \qquad \text{(II.15.2)}$$

Now since δ^k_i is a tensor, so from (II.15.2) b^{kj} may be a tensor. But we are not sure about the tensor character of b^{kj} as a_{ij} is not an arbitrary tensor. Hence quotient law can not be applied directly.

Let

$$B_j = a_{ij} D^i \tag{II.15.3}$$

where D^i is an arbitrary tensor of type $(1,0)$. So by inner product B_j defined by (II.15.3) is an arbitrary tensor of type $(0,1)$. The existence of B_j is confirmed by Cramer's rule of solution of non-homogeneous system of linear equations as $|a_{ij}| \neq 0$. Now,

$$\begin{aligned} B_j b^{kj} &= a_{ij} D^i b^{kj} \\ &= D^i a_{ij} b^{kj} \\ &= D^i \delta_i^k \\ &= D^k \end{aligned}$$

and D^k is a tensor of type $(1,0)$. Hence by alternative form of quotient law, we conclude that b^{kj} is a contravariant tensor of type $(2,0)$. This tensor b^{ij} is called the reciprocal tensor of a_{ij} and is usually denoted by a^{ij}, *i.e.*,

$$a^{ij} = \frac{A^{ij}}{|a_{ij}|},$$

where A^{ij} is the cofactor of a_{ij} in $|a_{ij}|$.

If b^{ij} is the reciprocal tensor of a_{ij}, then natural question arises whether a_{ij} is the reciprocal tensor of b^{ij}. The answer is affirmative as shown by the following :

From (II.15.2) we get

$$|a_{ij}|\,|b^{kj}| = |\delta_i^k| = 1$$

and hence

$$|b^{kj}| \neq 0.$$

Let

$$c_{ij} = \frac{\text{cofactor of } b^{ij} \text{ in } |b^{ij}|}{|b^{ij}|}. \tag{II.15.4}$$

Then

$$c_{ij} b^{kj} = \delta_i^k.$$

Multiplying both sides by a_{kl} we get

$$c_{ij} b^{kj} a_{kl} = \delta_i^k a_{kl}$$

or,

$$c_{ij} \delta_l^j = a_{il}$$

or,

$$c_{il} = a_{il}$$

i.e.,

$$c_{ij} = a_{ij}. \tag{II.15.5}$$

Since c_{ij} is the reciprocal tensor of b^{ij}, it follows from (II.15.5) that a_{ij} is the reciprocal tensor of b^{ij}. Thus we have proved that if b^{ij} is the reciprocal tensor of

a_{ij}, then a_{ij} is the reciprocal tensor of b^{ij}. The tensors a_{ij} and b^{ij} of type $(0,2)$ and $(2,0)$ respectively are called mutually reciprocal tensors or mutually conjugate tensors if $a_{ij}b^{kj} = \delta_i^k$.

Note : The conjugate of a tensor a_{ij} is usually written as a^{ij} and

$$a_{ij}a^{ji} = \delta_j^j = n.$$

Note : While defining the reciprocal or conjugate of a tensor most authors usually prefer to start the discussion considering symmetric tensors only. But this is not true, in general.

There is no harm if we consider any tensor symmetric, skew symmetric of even order or any tensor which is neither symmetric nor skew-symmetric. The only factor we have to take care that determinant of the tensor is non-zero.

II.16 Relative Tensor

A set of n^{p+q} functions $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ of n coordinates in a coordinate system (x^i) are said to form the components of a relative tensor of order $p+q$ and weight w if they transform according to the following rule into another coordinate system $(\bar{x}^i)$ given by

$$\bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = J^w \frac{\partial \bar{x}^{i_1}}{\partial x^{r_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{r_2}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{r_p}} \frac{\partial x^{s_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{s_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{s_q}}{\partial \bar{x}^{j_q}} A^{r_1 r_2 \ldots r_p}_{s_1 s_2 \ldots s_q},$$

where J is the Jacobian of the transformation *i.e.*, $J = \left|\frac{\partial x}{\partial \bar{x}}\right|$.

A relative tensor of weight 1 is called a tensor density. A relative tensor of weight zero is called an absolute tensor. A relative tensor of order zero is called relative scalar. Thus if

$$\bar{Q} = Q \left|\frac{\partial x}{\partial \bar{x}}\right|^w,$$

then Q is called relative scalar of weight w, where Q is a scalar. A relative scalar of weight one is called scalar density and a relative scalar of weight zero is called absolute scalar.

Relative tensors of the same type and weight may be added, and the sum is a relative tensor of the same type and weight. Relative tensors may be multiplied, the weight of the product being the sum of the weights of tensors entering in the product. The operation of contraction on a relative tensor yields a relative tensor of the same weight as the original tensor.

II.17 Cross Product or Vector Product of Two Vectors

Let us start with two covariant vectors A_i and B_i of S_n. If we consider

$$C_{ij} = A_i B_j - A_j B_i, \tag{II.17.1}$$

then we have

$$C_{ij} = -C_{ji}. \tag{II.17.2}$$

Now, since A_i and B_i are two covariant vectors,

$$\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p$$

and

$$\bar{B}_j = \frac{\partial x^q}{\partial \bar{x}^j} B_q .$$

Hence

$$\bar{A}_i \bar{B}_j = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} A_p B_q,$$

from which it follows that $A_i B_j$ is a tensor of type $(0,2)$. Similarly $A_j B_i$ is a tensor of type $(0,2)$. Therefore, by virtue of (II.17.1) C_{ij} is a tensor of type $(0,2)$ which is skew-symmetric as follows from (II.17.2). This skew-symmetric tensor C_{ij} is called the cross product or the vector product of the vectors A_i and B_i.

II.18 Examples

Example 4. Show that δ^i_j is a mixed tensor of order two.

Let u^k be an arbitrary vector. Evidently,

$$\delta^i_j u^j = u_i, \tag{II.18.1}$$

is a tensor of order 1. Thus, the inner product of δ^i_j with an arbitrary vector is a tensor. Hence by quotient law, δ^i_j is a mixed tensor of order two.

Example 5. The components A_i of a covariant vector in the coordinate system (x^i) are taken as follows : $A_1 = f$, where f is an arbitrary function of the x^i's and $A_j = 0, \quad j = 2, 3, \ldots n$. Show that the components in any other system $(\bar{x}^i)$ are given by

$$\bar{A}_i = f \frac{\partial x^1}{\partial \bar{x}^i}, \quad i = 1, 2, \ldots, n.$$

Since A_i are the components of a covariant vector, by transformation law we have

$$\begin{aligned} \bar{A}_i &= \frac{\partial x^j}{\partial \bar{x}^i} A_j, \quad i, j = 1, 2, \ldots n \\ &= \frac{\partial x^1}{\partial \bar{x}^i} A_1 + \frac{\partial x^2}{\partial \bar{x}^i} A_2 + \ldots + \frac{\partial x^n}{\partial \bar{x}^i} A_n. \end{aligned} \tag{II.18.2}$$

According to the problem, we have

$$A_1 = f, A_2 = A_3 = \ldots = A_n = 0$$

Hence (II.18.2) takes the form

$$\bar{A}_i = \frac{\partial x^1}{\partial \bar{x}^i} A_1 = f \frac{\partial x^1}{\partial \bar{x}^i}, \quad i = 1, 2, \ldots, n.$$

Example 6. If the relation $a^i_j v^j = 0$ holds for an arbitrary contravariant vector v^j, show that $a^i_j = 0$.

The given relation can be written as

$$a^i_1 v^1 + a^i_2 v^2 + \ldots + a^i_n v^n = 0, \quad i = 1, 2, \ldots, n. \tag{II.18.3}$$

Since v^j is arbitrary, we may choose it as $(1, 0, \ldots, 0)$. Then $v^1 = 1, v^2 = 0, \ldots, v^n = 0$. Hence from (II.18.3) we get

$$a^i_1 = 0. \tag{II.18.4}$$

Next, if we take v^j as $(0, 1, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$ in succession, then from (II.18.3) we can obtain

$$a^i_2 = 0, a^i_3 = 0, \ldots, a^i_n = 0. \tag{II.18.5}$$

From (II.18.4) and (II.18.5), it follows that

$$a^i_j = 0.$$

Example 7. If the relation $b^{ij} u_i u_j = 0$ holds for any arbitrary covariant vector u_i, prove that $b^{ij} + b^{ji} = 0$.

The given relation is

$$b^{ij} u_i u_j = 0,$$

where u_i is an arbitrary covariant vector. Since the number of components of the system b^{ij} is n^2, the left hand side of the given relation contains n^2 terms. Out of these n^2 terms in n number of terms the indices i and j are equal *i.e.*, the terms are $b^{11} u_1 u_1, b^{22} u_2 u_2, \ldots, b^{nn} u_n u_n$. Hence in the remaining $n(n-1)$ terms i and j are unequal, *i.e.*, the terms are like $b^{12} u_1 u_2, b^{13} u_1 u_3$ etc. Thus the given relation can be written as

$$b^{11} u_1 u_1 + b^{22} u_2 u_2 + \ldots + b^{nn} u_n u_n + b^{12} u_1 u_2 + \ldots = 0. \tag{II.18.6}$$

Since u_i is arbitrary we can choose u_i as $(1, 0, \ldots, 0)$. So $u_1 = 1, u_2 = 0, \ldots, u_n = 0$. Thus (II.18.6) takes the form

$$b^{11} = 0. \tag{II.18.7}$$

Next choosing u_i as $(0, 1, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$ in succession, from (II.18.6) we get

$$b^{22} = 0, b^{33} = 0, \ldots, b^{nn} = 0. \tag{II.18.8}$$

From (II.18.7) and (II.18.8) we get

$$b^{ij} = 0, \quad \text{for } i = j. \tag{II.18.9}$$

We now choose u_i as $(1,1,0,\ldots,0)$. So $u_1 = 1, u_2 = 1, u_3 = 0, \ldots, u_n = 0$. Then from (II.18.6) we get

$$b^{12} + b^{21} = 0 \qquad \text{by (II.18.9).}$$

Next we choose u_i as $(1,0,1,0,\ldots,0)$ etc. in succession such that in each case two components being 1 and the remaining being zero. Then from (II.18.6) we get by virtue of (II.18.9)

$$b^{13} + b^{31} = 0, b^{14} + b^{41} = 0 \quad \text{etc.}$$

Thus we have

$$b^{ij} + b^{ji} = 0 \qquad \text{for } i \neq j. \tag{II.18.10}$$

Combining (II.18.9) and (II.18.10) we get

$$b^{ij} + b^{ji} = 0. \tag{II.18.11}$$

Note : In particular, if b^{ij} is symmetric, then we have $b^{ij} = b^{ji}$ for all i, j and hence from (II.18.11) we get $b^{ij} = 0$ for all i, j.

Example 8. If $a^{ij}u_iu_j = b^{ij}u_iu_j$ for an arbitrary covariant vector u_i, show that

$$a^{ij} + a^{ji} = b^{ij} + b^{ji}.$$

The given relation can be written as

$$c^{ij}u_iu_j = 0 \tag{II.18.12}$$

where

$$c^{ij} = a^{ij} - b^{ij}. \tag{II.18.13}$$

By **Example 7**, it follows from (II.18.12) that

$$\begin{aligned} c^{ij} + c^{ji} &= 0, \\ \text{i.e.,} \quad a^{ij} - b^{ij} + a^{ji} - b^{ji} &= 0, \\ \text{or,} \quad a^{ij} + a^{ji} &= b^{ij} + b^{ji} \qquad \text{for all } i, j. \end{aligned}$$

Example 9. If A_{ij} is a symmetric tensor and $B_{ij} = A_{ji}$, show that B_{ij} is a symmetric tensor.

Since A_{ij} is symmetric, we have

$$A_{ij} = A_{ji} \qquad \text{for all } i, j.$$

Now we have

$$B_{ij} = A_{ji} = A_{ij}$$

i.e.,

$$B_{ij} = A_{ij} \qquad \text{for all } i, j.$$

Again,

$$\begin{aligned} & B_{ji} = A_{ij} \\ \text{and hence} \quad & B_{ij} = A_{ij} = B_{ji} \\ \text{i.e.,} \quad & B_{ij} = B_{ji} \qquad \text{for all } i, j. \end{aligned}$$

Hence B_{ij} is symmetric.

Example 10. If a tensor A_{ijkl} is symmetric in the first two indices from the left and skew-symmetric in the second and the fourth indices from the left, show that

$$A_{ijkl} = 0.$$

Since A_{ijkl} is symmetric with respect to i and j we have

$$A_{ijkl} = A_{jikl}\,. \tag{II.18.14}$$

Again, since A_{ijkl} is skew-symmetric with respect to j and l we have

$$A_{ijkl} = -A_{ilkj}\,. \tag{II.18.15}$$

Adding (II.18.14) and (II.18.15) we have

$$\begin{aligned} 2A_{ijkl} &= A_{jikl} - A_{ilkj} \\ &= -A_{jlki} - A_{ilkj} \quad \text{[by the property of skew-symmetry]} \\ &= -A_{jlki} - A_{likj} \quad \text{[by the property of symmetry]} \\ &= -A_{ljki} - (-A_{ljki}) \quad \text{[by the property of symmetry and skew-symmetry]} \\ &= -A_{ljki} + A_{ljki} \\ &= 0 \end{aligned}$$

Hence

$$A_{ijkl} = 0.$$

Example 11. If A^i and B^j are two contravariant vectors, then prove that their outer product A^iB^j is a tensor of type $(2,0)$. But the converse is not true. That is, a $(2,0)$ tensor can not be expressed as the outer product of two contravariant vectors.

From the transformation law of covariant vectors, we have

$$\begin{aligned} \bar{A}^i &= \frac{\partial \bar{x}^i}{\partial x^k} A^k \\ \text{and} \quad \bar{B}^j &= \frac{\partial \bar{x}^j}{\partial x^l} B^l. \end{aligned}$$

Taking outer multiplication of the above relations we get

$$\bar{A}^i \bar{B}^j = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} A^k B^l.$$

If we take $P^{kl} = A^k B^l$ and $\bar{P}^{kl} = \bar{A}^k \bar{B}^l$, then the last relation can be written as

$$\bar{P}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} P^{kl},$$

which implies that P^{kl} is a tensor of type $(2,0)$.

For the converse part, let us consider the 2-dimensional Euclidean space E_2. In E_2 let us take a $(2,0)$ tensor A^{ij} defined by

$$\begin{aligned} A^{ij} &= 1, \quad \text{if} \quad i = j \\ &= 0 \quad \text{if} \quad i \neq j. \end{aligned}$$

Let, if possible, there exist two contravariant vectors C^i and D^j such that A^{ij} can be written as an outer product of these two vectors, *i.e.*, $A^{ij} = C^i D^j$. Then

$$\begin{aligned} C^1 D^1 &= A^{11} = 1 \quad \text{implies} \quad C^1 \neq 0, \\ C^1 D^2 &= A^{12} = 0 \quad \text{implies} \quad D^2 = 0, \quad \text{since} \quad C^1 \neq 0. \end{aligned}$$

Again $C^2 D^2 = A^{22} = 1$, but since $D^2 = 0$, therefore $C^2 D^2$ must be 0, *i.e.*, $A^{22} = 0$, which is a contradiction. Hence, A^{ij} can not be written as the outer product of two contravariant vectors.

Example 12. If the equality $\lambda^i_j A^j = \sigma A^i$ holds for any contravariant vector A^i, where σ is a scalar, show that

$$\lambda^i_j = \sigma \delta^i_j.$$

The given equality can be written as

$$\lambda^i_j A^j = \sigma \delta^i_j A^j \tag{II.18.16}$$

$$\text{or,} \quad (\lambda^i_j - \sigma \delta^i_j) A^j = 0 \tag{II.18.17}$$

$$\text{or,} \quad \mu^i_j A^j = 0 \tag{II.18.18}$$

$$\text{where,} \quad \mu^i_j = \lambda^i_j - \sigma \delta^i_j. \tag{II.18.19}$$

But by **Example 6**, it follows from (II.18.18) that

$$\mu^i_j = 0 \qquad \text{for all } i, j$$

i.e.,

$$\lambda^i_j = \sigma \delta^i_j.$$

Example 13. If f is an invariant, determine whether $\frac{\partial^2 f}{\partial x^i \partial x^j}$ is a tensor.

Since f is an invariant, we have

$$\bar{f} = f.$$

Then,

$$\frac{\partial \bar{f}}{\partial \bar{x}^j} = \frac{\partial f}{\partial x^p}\frac{\partial x^p}{\partial \bar{x}^j}, \quad \text{since} \quad \bar{f} = f.$$

So,

$$\begin{aligned}\frac{\partial^2 \bar{f}}{\partial \bar{x}^i \partial \bar{x}^j} &= \frac{\partial^2 f}{\partial x^l \partial x^p}\frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^p}{\partial \bar{x}^j} + \frac{\partial f}{\partial x^p}\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} \\ &= \frac{\partial^2 f}{\partial x^l \partial x^p}\frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^p}{\partial \bar{x}^j} + \frac{\partial f}{\partial x^p}\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}.\end{aligned}$$

Due to the presence of the second term in the right hand side, $\frac{\partial^2 f}{\partial x^i \partial x^j}$ is not a tensor in general.

Example 14. If

$$\begin{aligned}\delta_{ij} &= 1 \qquad \text{when } i = j \\ &= 0 \qquad \text{when } i \neq j,\end{aligned}$$

for all coordinate systems (x^i), determine whether δ_{ij} are the components of a covariant tensor of second order.

Assuming that δ_{ij} is a tensor, we have, for its components in a changed (primed) coordinate systems, the values

$$\begin{aligned}\bar{\delta}_{ij} &= \frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial x^\beta}{\partial \bar{x}^j}\delta_{\alpha\beta} \\ &= \sum_\alpha \frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial x^\alpha}{\partial \bar{x}^j} \qquad (\text{since } \delta_{\alpha\alpha} = 1 \quad \text{and} \quad \delta_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta) \\ &\neq \delta_{ij}\end{aligned}$$

in general. This implies that δ_{ij} are not the components of a covariant tensor of order 2. (However, if the transformations are orthogonal, it is easy to see that $\bar{\delta}_{ij} = \delta_{ij}$. Thus the δ_{ij} does form components of a covariant tensor for orthogonal transformations.)

Note : The components

$$\begin{aligned}\delta^i_j &= 1 \qquad \text{when } i = j \\ &= 0 \qquad \text{when } i \neq j,\end{aligned}$$

form a mixed tensor of order $(1, 1)$.

Example 15. If $a^{ij}u_iu_j$ is an invariant for an arbitrary covariant vector u_i, show that $a^{ij} + a^{ji}$ is a tensor.

Since $a^{ij}u_iu_j$ is an invariant, we have

$$\begin{aligned}\bar{a}^{ij}\bar{u}_i\bar{u}_j &= a^{ij}u_iu_j \\ &= a^{ij}\frac{\partial \bar{x}^p}{\partial x^i}\bar{u}_p\frac{\partial \bar{x}^q}{\partial x^j}\bar{u}_q\end{aligned}$$

$$= a^{ij}\frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j}\bar{u}_p\bar{u}_q$$

$$= a^{pq}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\bar{u}_i\bar{u}_j,$$

interchanging the dummy indices i and j by p and q respectively,

or, $$\left(\bar{a}^{ij} - a^{pq}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\right)\bar{u}_i\bar{u}_j = 0,$$

or, $$\bar{B}^{ij}\bar{u}_i\bar{u}_j = 0 \tag{II.18.20}$$

where,

$$\bar{B}^{ij} = \bar{a}^{ij} - a^{pq}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}. \tag{II.18.21}$$

Since $\bar{u}_i$ is arbitrary, it follows from (II.18.20) that

$$\bar{B}^{ij} + \bar{B}^{ji} = 0. \tag{II.18.22}$$

In virtue of (II.18.21), the relation (II.18.22) can be expressed as

$$\bar{a}^{ij} - a^{pq}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q} + \bar{a}^{ji} - a^{pq}\frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^i}{\partial x^q} = 0$$

or,
$$\bar{a}^{ij} + \bar{a}^{ji} = a^{pq}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q} + a^{pq}\frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^i}{\partial x^q}$$

$$= a^{pq}\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q} + a^{qp}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial \bar{x}^i}{\partial x^p}$$

(replacing the dummy indices q by p and p by q in the second term of r.h.s.),

$$= (a^{pq} + a^{qp})\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}. \tag{II.18.23}$$

From (II.18.23) it follows that $a^{ij} + a^{ji}$ is a contravariant tensor of order 2.

Example 16. Show that in an n-dimensional space a covariant skew-symmetric tensor of second order has at most $\frac{1}{2}n(n-1)$ different arithmetical components.

Since a covariant tensor a_{ij} of second order has n^2 components in S_n, they may be given as follows :

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}. \end{array}$$

Since a_{ij} is skew-symmetric, we have

$$a_{ij} = -a_{ji}, \qquad i, j = 1, 2, \ldots, n.$$

Hence

$$a_{ii} = 0 \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

Thus

$$a_{11} = a_{22} = \ldots = a_{nn} = 0.$$

So the number of independent components corresponding to a repeated suffix is zero. Hence the number of components of distinct suffixes $= n^2 - n$. But due to the skew-symmetry, this number is reduced to $\frac{n^2-n}{2}$. Therefore the total number of independent components of a_{ij} is $\frac{n^2-n}{2} + 0 = \frac{1}{2}n(n-1)$.

Example 17. If a_{ijk} is completely skew-symmetric and the indices run from 1 to n, show that the number of distinct non-vanishing components of a_{ijk} is $\frac{n(n-1)(n-2)}{6}$.

Since a_{ijk} is completely skew-symmetric, we have

$$a_{ijk} = -a_{jik}, \tag{II.18.24}$$

$$a_{ijk} = -a_{kji}, \tag{II.18.25}$$

$$a_{ijk} = -a_{ikj}. \tag{II.18.26}$$

When a_{ijk} is of type a_{iii}, then taking $i = j = k$ in (II.18.24) we get

$$a_{iii} = 0 \qquad \text{for all } i = 1, 2, \ldots, n.$$

Hence the number of independent components of a_{ijk} is zero for $i = j = k$.

When a_{ijk} is of the type a_{iik}, then by skew-symmetry

$$a_{iik} = -a_{iik}$$

$$i.e., \qquad a_{iik} = 0 \qquad \text{for all } i, k = 1, 2, \ldots, n.$$

Hence the number of independent components of a_{ijk} is again zero for $i = j$; $i, j, k = 1, 2, \ldots, n$. Therefore, the number of distinct non-vanishing components of a_{ijk} is ${}^nC_3 = \frac{n(n-1)(n-2)}{6}$.

Example 18. Prove that $\frac{\partial A_i}{\partial x^j}$ is not a tensor though A_i is a tensor of type $(0, 1)$.

Since A_i is a covariant tensor, we have

$$\bar{A}_i = \frac{\partial x^k}{\partial \bar{x}^i} A_k. \tag{II.18.27}$$

Differentiating both sides of (II.18.27) with respect to $\bar{x}^j$ we get

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial A_k}{\partial x^l} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^k}{\partial \bar{x}^i} + A_k \frac{\partial^2 x^k}{\partial \bar{x}^j \partial \bar{x}^i}. \tag{II.18.28}$$

Due to the presence of second term on r.h.s. of (II.18.28), it follows that $\frac{\partial A_i}{\partial x^j}$ is not a tensor.

Example 19. If the tensors a_{ij} and b_{ij} are symmetric and u^i, v^i are components of contravariant vectors satisfying the equations

$$(a_{ij} - \alpha b_{ij})\, u^i = 0,$$

$$(a_{ij} - \beta b_{ij})\, v^i = 0, \qquad i, j = 1, 2, \ldots, n, \qquad \alpha \neq \beta,$$

prove that

$$b_{ij} u^i v^j = 0, \qquad a_{ij} u^i v^j = 0.$$

We have

$$(a_{ij} - \alpha b_{ij})\, u^i = 0, \tag{II.18.29}$$
$$(a_{ij} - \beta b_{ij})\, v^i = 0. \tag{II.18.30}$$

Multiplying (II.18.29) and (II.18.30) respectively by v^j and u^j and subtracting we get

$$a_{ij} u^i v^j - a_{ij} v^i u^j - \alpha b_{ij} u^i v^j + \beta b_{ij} v^i u^j = 0.$$

Interchanging i and j in the second and fourth terms and noting that

$$b_{ij} = b_{ji}, \qquad a_{ij} = a_{ji},$$

we get

$$a_{ij} u^i v^j - a_{ij} v^j u^i - \alpha b_{ij} u^i v^j + \beta b_{ij} v^j u^i = 0,$$
$$\text{or,} \qquad -(\alpha - \beta)\, b_{ij} u^i v^j = 0.$$

Since $\alpha \neq \beta$, it follows that

$$b_{ij} u^i v^j = 0. \tag{II.18.31}$$

Again multiplying (II.18.29) by v^j and remembering (II.18.31) we obtain

$$a_{ij} u^i v^j - \alpha 0 = 0,$$
$$\text{or,} \qquad a_{ij} u^i v^j = 0.$$

Example 20. If a_{ij} and a^{ji} are reciprocal symmetric tensors of the second order, show that

$$(i) \qquad a^{ij} \frac{\partial a_{ij}}{\partial x^k} + a_{ij} \frac{\partial a^{ij}}{\partial x^k} = 0,$$
$$(ii) \qquad \frac{\partial \log a}{\partial x^k} = a^{ij} \frac{\partial a_{ij}}{\partial x^k} = -a_{ij} \frac{\partial a^{ij}}{\partial x^k},$$

where $a = |a_{ij}|$.

Since a_{ij} and a^{ji} are reciprocal tensors, we have

$$a_{ij} a^{ji} = n.$$

Differentiating the above relation with respect to x^k, we get

$$a^{ij} \frac{\partial a_{ij}}{\partial x^k} + a_{ij} \frac{\partial a^{ij}}{\partial x^k} = 0$$

which is the result (i). From this we have

$$a^{ij}\frac{\partial a_{ij}}{\partial x^k} = -a_{ij}\frac{\partial a^{ij}}{\partial x^k}. \qquad \text{(II.18.32)}$$

Now,

$$\frac{1}{a}\frac{\partial a}{\partial x^k} = \frac{\text{cofactor of } a_{ij} \text{ in } |a_{ij}|}{a}\left(\frac{\partial a_{ij}}{\partial x^k}\right)$$

$$= a^{ij}\frac{\partial a_{ij}}{\partial x^k}, \text{ by definition of } a^{ij}$$

or,
$$\frac{\partial \log a}{\partial x^k} = a^{ij}\frac{\partial a_{ij}}{\partial x^k}.$$

Combining this result with (II.18.32) we get

$$\frac{\partial \log a}{\partial x^k} = -a_{ij}\frac{\partial a^{ij}}{\partial x^k},$$

which is the result (ii).

Example 21. If f is a scalar function of coordinates x^i, then show that $\frac{\partial f}{\partial x^i}$ is a covariant vector.

Let us consider the coordinate transformation

$$x^i \to \bar{x}^i$$

i.e., $$\bar{x}^i = \bar{x}^i\left(x^i\right).$$

Then

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^i},$$

which is of the type $\bar{A}_i = A_k\frac{\partial x^k}{\partial \bar{x}^i}$, where $A_i = \frac{\partial f}{\partial x^i}$.

This is the law of transformation of a covariant vector. Hence $\frac{\partial f}{\partial x^i}$ is a covariant vector.

Example 22. If the relation $A_{ijk}u^iu^ju^k = 0$ holds for any arbitrary contravariant vector u^i, show that

$$A_{ijk} + A_{jki} + A_{kij} + A_{jik} + A_{kji} + A_{ikj} = 0,$$

where it is given that A_{ijk} are constants for all i, j, k.

Let

$$A = A_{ijk}u^iu^ju^k = 0.$$

This implies

$$A = A_{pqr}u^pu^qu^r = 0.$$

Since u^p is arbitrary contravariant vector, we obtain

$$\frac{\partial A}{\partial u^i} = A_{iqr}u^qu^r + A_{pir}u^pu^r + A_{pqi}u^pu^q = 0,$$

$$\frac{\partial^2 A}{\partial u^j \partial u^i} = A_{ijr}u^r + A_{iqj}u^q + A_{jir}u^r + A_{pij}u^p + A_{jqi}u^q + A_{pji}u^p = 0,$$

$$\frac{\partial^3 A}{\partial u^k \partial u^j \partial u^i} = A_{ijk} + A_{ikj} + A_{jik} + A_{kij} + A_{jki} + A_{kji} = 0.$$

Hence

$$A_{ijk} + A_{ikj} + A_{jik} + A_{kij} + A_{jki} + A_{kji} = 0.$$

Example 23. If the relation $A_{ijk}u^i u^j u^k = 0$ holds for any arbitrary contravariant vector u^i, where A_{ijk} is a tensor symmetric in i and j , show that

$$A_{ijk} + A_{jki} + A_{kij} = 0.$$

From **Example 22**, we have

$$A_{ijk} + A_{ikj} + A_{jik} + A_{kij} + A_{jki} + A_{kji} = 0. \qquad \text{(II.18.33)}$$

Since A_{ijk} is symmetric in i and j we have

$$A_{ijk} = A_{jik}, \qquad \text{(II.18.34)}$$

$$A_{ikj} = A_{jki} \qquad \text{(II.18.35)}$$

and

$$A_{kij} = A_{kji}. \qquad \text{(II.18.36)}$$

Using (II.18.34), (II.18.35) and (II.18.36) in (II.18.33) we get

$$2A_{ijk} + 2A_{jki} + 2A_{kij} = 0$$

i.e.,

$$A_{ijk} + A_{jki} + A_{kij} = 0.$$

Example 24. If the relation $A_{ijkl}u^i v^j u^k v^l = 0$ holds for arbitrary contravariant vectors u^i and v^i, prove that

$$A_{ijkl} + A_{ilkj} + A_{kjil} + A_{klij} = 0.$$

Let

$$A = A_{pqrs}u^p v^q u^r v^s = 0.$$

Since u^i and v^i are arbitrary contravariant vectors, we get from the above

$$\frac{\partial A}{\partial u^i} = A_{iqrs}v^q u^r v^s + A_{pqis}u^p v^q v^s = 0,$$

$$\Rightarrow \frac{\partial^2 A}{\partial v^j \partial u^i} = A_{ijrs}u^r v^s + A_{iqrj}v^q u^r + A_{pjis}u^p v^s + A_{pqij}u^p v^q = 0,$$

$$\Rightarrow \frac{\partial^3 A}{\partial u^k \partial v^j \partial u^i} = A_{ijks}v^s + A_{iqkj}v^q + A_{kjis}v^s + A_{kqij}v^q = 0,$$

$$\Rightarrow \frac{\partial^4 A}{\partial v^l \partial u^k \partial v^j \partial u^i} = A_{ijkl} + A_{ilkj} + A_{kjil} + A_{klij} = 0.$$

Example 25. If B^{ij} are the components of a contravariant tensor and C_i, D_i are the components of two covariant vectors, show that $B^{ij}C_iD_j$ is an invariant.

Since B^{ij} are the components of a contravariant tensor and C_i and D_i are the components of two covariant vectors, we have

$$\bar{B}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^p}B^{kp}, \tag{II.18.37}$$

$$\bar{C}_i = \frac{\partial x^r}{\partial \bar{x}^i}C_r \tag{II.18.38}$$

and

$$\bar{D}_j = \frac{\partial x^s}{\partial \bar{x}^j}D_s. \tag{II.18.39}$$

Now,

$$\begin{aligned}
\bar{B}^{ij}\bar{C}_i\bar{D}_j &= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}B^{kp}C_rD_s \qquad \text{[using (II.18.37)-(II.18.39)]} \\
&= \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial x^s}{\partial \bar{x}^j}B^{kp}C_rD_s \\
&= \delta^r_k\delta^s_pB^{kp}C_rD_s \\
&= \delta^r_kB^{kp}\delta^s_pD_sC_r \\
&= B^{rp}C_rD_p \\
&= B^{ij}C_iD_j \qquad \text{[replacing the dummy indices } r \text{ and } p \\
&\qquad\qquad \text{by } i \text{ and } j \text{ respectively]}.
\end{aligned}$$

Hence $B^{ij}C_iD_j$ is an invariant.

Example 26. If a_{ij}, b_k are the components of a symmetric covariant tensor and a covariant vector respectively satisfying the relation $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$, prove that either $a_{ij} = 0$ or $b_k = 0$.

We first suppose that one of the b_k's, say b_1 is not zero. Then putting $i = j = k = 1$ in the given relation we get

$$\begin{aligned}
a_{11}b_1 + a_{11}b_1 + a_{11}b_1 &= 0 \\
\text{or,} \qquad 3a_{11}b_1 &= 0
\end{aligned}$$

$$\text{or,} \qquad a_{11} = 0 \qquad (\text{since} \quad b_1 \neq 0). \tag{II.18.40}$$

Next putting $i = j = 1$ in the given relation, we get

$$a_{11}b_k + a_{1k}b_1 + a_{k1}b_1 = 0.$$

By virtue of (II,18.40) and the symmetry of a_{ij}, we obtain from the above relation

$$2a_{1k}b_1 = 0$$

i.e.,

$$a_{1k} = 0 \qquad \text{for all } k. \tag{II.18.41}$$

Finally, putting $k = 1$ in the given relation, we get

$$\begin{aligned} & a_{ij}b_1 + a_{j1}b_i + a_{1i}b_j = 0 \\ \text{or,} \quad & a_{ij}b_1 + a_{1j}b_i + a_{1i}b_j = 0 \\ \text{or,} \quad & a_{ij}b_1 = 0 \qquad [\text{using (II.18.41)}] \\ \text{or,} \quad & a_{ij} = 0 \qquad (\text{since, } b_1 \neq 0). \end{aligned} \tag{II.18.42}$$

Next, we suppose that one of the components of a_{ij}, say $a_{11} \neq 0$. Then b_k must be zero for all k, otherwise a_{ij} must be zero as already proved. Hence either $a_{ij} = 0$ or $b_k = 0$.

Example 27. If a_{ij}, b_{kl} are the components of two symmetric tensors in S_n such that

$$|b_{kl}| \neq 0 \text{ and } a_{ij}b_{kl} - a_{il}b_{jk} + a_{jk}b_{il} - a_{kl}b_{ij} = 0$$

then prove that, $a_{ij} = \alpha b_{ij}$, where α is some scalar.

Since b_{kl} is symmetric and $|b_{kl}| \neq 0$, we can obtain c^{ij} such that

$$b_{ij}c^{ik} = \delta^k_j. \tag{II.18.43}$$

Multiplying both sides of the given relation by c^{kl} we get

$$\begin{aligned} a_{ij}c^{kl}b_{kl} - a_{il}c^{kl}b_{jk} + a_{jk}c^{kl}b_{il} - a_{kl}c^{kl}b_{ij} &= 0 \\ \text{or, } na_{ij} - a_{il}\delta^l_j + a_{jk}\delta^k_i - \sigma b_{ij} &= 0 \end{aligned}$$

where $\sigma = a_{kl}c^{kl}$. Therefore

$$\begin{aligned} na_{ij} - a_{ij} + a_{ji} - \sigma b_{ij} &= 0 \\ \text{or,} \quad a_{ij} &= \alpha b_{ij}, \end{aligned}$$

where $\alpha = \frac{\sigma}{n}$ is a scalar because σ is so.

Example 28. If a_{ij} is a covariant tensor such that $|a_{ij}| \neq 0$, examine whether the determinant $|a_{ij}|$ is an invariant.

Since a_{ij} is a covariant tensor, we have

$$\bar{a}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}a_{pq}.$$

Since covariant and contravariant transformations of rank 2 tensors can be expressed in matrix multiplication, therefore

$$\left(\bar{a}_{ij}\right) = \left(\frac{\partial x^p}{\partial \bar{x}^i}\right)\left(\frac{\partial x^q}{\partial \bar{x}^j}\right)\left(a_{pq}\right).$$

Hence

$$|\bar{a}_{ij}| = \left|\frac{\partial x^p}{\partial \bar{x}^i}\right|\left|\frac{\partial x^q}{\partial \bar{x}^j}\right||a_{pq}|$$

$$= \left|\frac{\partial x^p}{\partial \bar{x}^i}\right|^2 |a_{pq}|$$
$$= J^2 |a_{pq}| \qquad \text{(II.18.44)}$$

where J is the Jacobian of the transformation $\left|\frac{\partial x}{\partial \bar{x}}\right|$. From (II.18.44), it follows that $|a_{ij}|$ is not, in general, an invariant due to the presence of the term J^2 in the right hand side.

Example 29. If the relation $A^i_{jk} u^{kl}_i = 0$ holds for an arbitrary tensor u^{kl}_i, show that

$$A^i_{jk} = 0.$$

The given relation can be written as

$$A^1_{j1} u^{1l}_1 + A^2_{j2} u^{2l}_2 + \ldots + A^n_{jn} u^{nl}_n = 0. \qquad \text{(II.18.45)}$$

For $l = 1$, we get from (II.18.45)

$$A^1_{j1} u^{11}_1 + A^2_{j2} u^{21}_2 + \ldots + A^n_{jn} u^{n1}_n = 0. \qquad \text{(II.18.46)}$$

Since u^{ij}_k is an arbitrary tensor, we may choose its components arbitrarily. First we take

$$u^{11}_1 = 1, \quad u^{21}_2 = 0, \quad \ldots, \quad u^{n1}_n = 0 \ .$$

Then from (II.18.46) we have

$$A^1_{j1} = 0. \qquad \text{(II.18.47)}$$

Similarly, if we take

$$u^{11}_1 = 0, \quad u^{21}_2 = 1, \quad \ldots, \quad u^{n1}_n = 0,$$

we may obtain from (II.18.46) that

$$A^2_{j2} = 0. \qquad \text{(II.18.48)}$$

Proceeding in this way we may obtain

$$A^r_{jr} = 0 \text{ for all } r \text{ (summation not implied).} \qquad \text{(II.18.49)}$$

Combining (II.18.47), (II.18.48) and (II.18.49) we get

$$A^i_{jk} = 0 \qquad \text{for} \qquad i = k. \qquad \text{(II.18.50)}$$

Next, if $i \neq k$, we get from the given relation

$$A^1_{j2} u^{2l}_1 + A^2_{j1} u^{1l}_2 + \ldots = 0.$$

In a similar manner, choosing u^{ij}_k arbitrarily, we can show that

$$A^i_{jk} = 0 \qquad \text{for} \qquad i \neq k. \qquad \text{(II.18.51)}$$

Combining (II.18.50) and (II.18.51) we get

$$A^i_{jk} = 0 \qquad \text{for all} \qquad i, j, k.$$

Example 30. If $a_{ij}u^iu^j$ is an invariant, where u^i is an arbitrary contravariant vector, a_{ij} is a symmetric tensor and $u^i = A^i + B^i$, then show that $a_{ij}A^iB^j$ is an invariant.

Since $a_{ij}u^iu^j$ is an invariant, we have

$$\bar{a}_{ij}\bar{u}^i\bar{u}^j = a_{ij}u^iu^j \ . \tag{II.18.52}$$

Since $u^i = A^i + B^i$ and a_{ij} is symmetric, we obtain from (II.18.52)

$$\bar{a}_{ij}\bar{A}^i\bar{A}^j + \bar{a}_{ij}\bar{B}^i\bar{B}^j + 2\bar{a}_{ij}\bar{A}^i\bar{B}^j = a_{ij}A^iA^j + a_{ij}B^iB^j + 2a_{ij}A^iB^j. \tag{II.18.53}$$

Since u^i is an arbitrary contravariant vector, we get from (II.18.52)

$$\bar{a}_{ij}\bar{A}^i\bar{A}^j = a_{ij}A^iA^j \tag{II.18.54}$$

and

$$\bar{a}_{ij}\bar{B}^i\bar{B}^j = a_{ij}B^iB^j. \tag{II.18.55}$$

Using (II.18.54) and (II.18.55) in (II.18.53) we obtain

$$\bar{a}_{ij}\bar{A}^i\bar{B}^j = a_{ij}A^iB^j.$$

Example 31. Prove that the transformations of covariant vectors form a group.

Let S be the set of all transformations of covariant vectors and T, T' be two transformations from the system (x^i) to the system $(\bar{x}^i)$ and from $(\bar{x}^i)$ to $\bar{\bar{x}}^i$ given by

$$T \ : \ \bar{A}_i = A_p\frac{\partial x^p}{\partial \bar{x}^i} \tag{II.18.56}$$

$$T' \ : \ \bar{\bar{A}}_i = \bar{A}_r\frac{\partial \bar{x}^r}{\partial \bar{\bar{x}}^i}. \tag{II.18.57}$$

Then the product transformation $T'T$ is given by

$$T'T : \bar{\bar{A}}_i = A_p\frac{\partial x^p}{\partial \bar{x}^r}\frac{\partial \bar{x}^r}{\partial \bar{\bar{x}}^i} \qquad [\text{ by the above two equations}]$$

$$= A_p\frac{\partial x^p}{\partial \bar{\bar{x}}^i}. \tag{II.18.58}$$

From (II.18.58), it follows that

$$T'T \in S. \tag{II.18.59}$$

Let E be the transformation given by

$$E : A_i = A_i\frac{\partial x^p}{\partial x^p}. \tag{II.18.60}$$

Then

$$ET = TE = T.$$

Hence E is the identity transformation and from (II.18.60) it follows that

$$E \in S. \tag{II.18.61}$$

From (II.18.56) we get

$$A_p = \bar{A}_i \frac{\partial \bar{x}^i}{\partial x^p}. \tag{II.18.62}$$

We denote this by T_1. Since T_1 represents the transformation from the system $(\bar{x}^i)$ to the system (x^i), it is the inverse of T. Hence from (II.18.62) it follows that

$$T_1 \in S \tag{II.18.63}$$

Finally, it can be easily verified that if T'' represents a transformation from the system (x^i) to the system $(\bar{\bar{x}}^i)$, then

$$(T''T')\,T = T''\,(T'T)\,. \tag{II.18.64}$$

In virtue of (II.18.59), (II.18.61), (II.18.63) and (II.18.64), it follows that S forms a group.

Example 32. If a vector has components $\dot{x}$, $\dot{y}$ in rectangular Cartesian coordinates then show that $\dot{r}$, $\dot{\theta}$ are components in polar coordinates and if a vector has components $\ddot{x}, \ddot{y}$ in Cartesian coordinates, then prove that its components in polar coordinates are $\ddot{r} - r\dot{\theta}^2$, $\dot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}$, where dots representing differentiation with respect to a parameter t.

Let us consider contravariant transformation

$$\bar{A}^i = A^p \frac{\partial \bar{x}^i}{\partial x^p}$$

$$\text{or,} \quad \bar{A}^i = A^1 \frac{\partial \bar{x}^i}{\partial x^1} + A^2 \frac{\partial \bar{x}^i}{\partial x^2}.$$

This implies,

$$\bar{A}^1 = A^1 \frac{\partial \bar{x}^1}{\partial x^1} + A^2 \frac{\partial \bar{x}^1}{\partial x^2}$$

$$\text{and } \bar{A}^2 = A^1 \frac{\partial \bar{x}^2}{\partial x^1} + A^2 \frac{\partial \bar{x}^2}{\partial x^2}.$$

Hence,

$$\bar{A}^1 = A^1 \frac{\partial r}{\partial x} + A^2 \frac{\partial r}{\partial y} \tag{II.18.65}$$

and

$$\bar{A}^2 = A^1 \frac{\partial \theta}{\partial x} + A^2 \frac{\partial \theta}{\partial y} \tag{II.18.66}$$

where $x^1 = x, \quad x^2 = y, \quad \bar{x}^1 = r, \quad \bar{x}^2 = \theta.$

We know that,

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then

$$x^2 + y^2 = r^2, \qquad \theta = \tan^{-1}\frac{y}{x}.$$

Hence

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial \theta}{\partial x} = \frac{-y}{r^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}. \tag{II.18.67}$$

Again

$$r^2 = x^2 + y^2 \quad \Rightarrow \qquad r\dot{r} = x\dot{x} + y\dot{y}. \tag{II.18.68}$$

Differentiating again with respect to t, we get

$$\dot{r}^2 + r\ddot{r} = \dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}$$

or,

$$x\ddot{x} + y\ddot{y} = \dot{r}^2 + r\ddot{r} - \left(\dot{x}^2 + \dot{y}^2\right). \tag{II.18.69}$$

Now

$$\begin{aligned} x = r\cos\theta \quad &\Rightarrow \qquad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\ y = r\sin\theta \quad &\Rightarrow \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \end{aligned}$$

Squaring and adding we get

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2. \tag{II.18.70}$$

Using (II.18.70) in (II.18.69) we get

$$x\ddot{x} + y\ddot{y} = \dot{r}^2 + r\ddot{r} - \left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

or,

$$x\ddot{x} + y\ddot{y} = r\ddot{r} - r^2\dot{\theta}^2. \tag{II.18.71}$$

Case 1 : We first suppose $A^1 = \dot{x}, \quad A^2 = \dot{y}$. Then we are to prove that

$$\bar{A}^1 = \dot{r}, \qquad \bar{A}^2 = \dot{\theta}.$$

In this case (II.18.65) and (II.18.66) become

$$\begin{aligned} \bar{A}^1 &= \dot{x}\frac{\partial r}{\partial x} + \dot{y}\frac{\partial r}{\partial y} = \dot{x}\frac{x}{r} + \dot{y}\frac{y}{r} \\ \bar{A}^2 &= \dot{x}\frac{\partial \theta}{\partial x} + \dot{y}\frac{\partial \theta}{\partial y} = \dot{x}\left(\frac{-y}{r^2}\right) + \dot{y}\left(\frac{x}{r^2}\right), \end{aligned}$$

or,

$$\bar{A}^1 = \frac{x\dot{x} + y\dot{y}}{r} = \frac{r\dot{r}}{r} = \dot{r}$$

and

$$\bar{A}^2 = \frac{-\dot{x}y + \dot{y}x}{r^2} = \dot{\theta}$$

for

$$\theta = \tan^{-1}\frac{y}{x} \quad \Rightarrow \quad \dot{\theta} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{\dot{y}x - \dot{x}y}{x^2}\right) = \frac{\dot{y}x - \dot{x}y}{r^2}. \tag{II.18.72}$$

Differentiating again with respect to t

$$\ddot{\theta} = \frac{\{(\ddot{y}x + \dot{y}\dot{x}) - (\ddot{x}y + \dot{x}\dot{y})\}r^2 - 2r\dot{r}(\dot{y}x - \dot{x}y)}{r^4}$$

$$\text{or,} \quad \ddot{\theta} = \left(\frac{\ddot{y}x - \ddot{x}y}{r^2}\right) - \frac{2\dot{r}}{r^3}(\dot{y}x - \dot{x}y)$$

$$= \frac{\ddot{y}x - \ddot{x}y}{r^2} - \frac{2\dot{r}}{r^3}r^2\dot{\theta} \quad \text{[by equation (II.18.72)]}$$

or,

$$\frac{\ddot{y}x - \ddot{x}y}{r^2} = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}. \tag{II.18.73}$$

Case 2 : Next we suppose $A^1 = \ddot{x}$, $A^2 = \ddot{y}$. In this case (II.18.65) and (II.18.66) become

$$\bar{A}^1 = \ddot{x}\frac{\partial r}{\partial x} + \ddot{y}\frac{\partial r}{\partial y} = \ddot{x}\frac{\dot{x}}{r} + \ddot{y}\frac{y}{r}, \tag{II.18.74}$$

$$\bar{A}^2 = \ddot{x}\frac{\partial \theta}{\partial x} + \ddot{y}\frac{\partial \theta}{\partial y} = \ddot{x}\left(\frac{-y}{r^2}\right) + \ddot{y}\left(\frac{x}{r^2}\right). \tag{II.18.75}$$

From (II.18.71) and (II.18.74)

$$\bar{A}^1 = \frac{1}{r}\left[r\ddot{r} - r^2\dot{\theta}^2\right]$$

or,

$$\bar{A}^1 = \ddot{r} - r\dot{\theta}^2. \tag{II.18.76}$$

From (II.18.73) and (II.18.75), we have

$$\bar{A}^2 = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}. \tag{II.18.77}$$

From (II.18.76) and (II.18.77) we get the required result.

Example 33. A covariant vector has components $xy, 2y - z^2, xz$ in rectangular Cartesian coordinates. Determine its components in spherical polar coordinates.

The relation between Cartesian coordinates and spherical polar coordinates is given by

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}. \tag{II.18.78}$$

Let us suppose that

$$\begin{aligned} x^1 &= x, \quad x^2 = y, \quad x^3 = z, \\ \bar{x}^1 &= r, \quad \bar{x}^2 = \theta, \quad \bar{x}^3 = \phi. \end{aligned} \tag{II.18.79}$$

Suppose

$$A_1 = xy, \qquad A_2 = 2y - z^2, \qquad A_3 = xz.$$

Then A_1, A_2, A_3 are components of a covariant tensor in (x^i) system. We now determine $\bar{A}^i$ in $(\bar{x}^i)$ system. By transformation law, we have

$$\begin{aligned} \bar{A}_i &= A_p \frac{\partial x^p}{\partial \bar{x}^i} = A_1 \frac{\partial x^1}{\partial \bar{x}^i} + A_2 \frac{\partial x^2}{\partial \bar{x}^i} + A_3 \frac{\partial x^3}{\partial \bar{x}^i} \\ \text{or,} \quad \bar{A}_i &= A_1 \frac{\partial x}{\partial \bar{x}^i} + A_2 \frac{\partial y}{\partial \bar{x}^i} + A_3 \frac{\partial z}{\partial \bar{x}^i}. \end{aligned} \tag{II.18.80}$$

Taking $i = 1$ in (II.18.80) we get

$$\begin{aligned} \bar{A}_1 &= A_1 \frac{\partial x}{\partial \bar{x}^1} + A_2 \frac{\partial y}{\partial \bar{x}^1} + A_3 \frac{\partial z}{\partial \bar{x}^1} \\ \text{or,} \quad \bar{A}_1 &= A_1 \frac{\partial x}{\partial r} + A_2 \frac{\partial y}{\partial r} + A_3 \frac{\partial z}{\partial r}. \end{aligned} \tag{II.18.81}$$

Similarly

$$\bar{A}_2 = A_1 \frac{\partial x}{\partial \theta} + A_2 \frac{\partial y}{\partial \theta} + A_3 \frac{\partial z}{\partial \theta}, \tag{II.18.82}$$

$$\bar{A}_3 = A_1 \frac{\partial x}{\partial \phi} + A_2 \frac{\partial y}{\partial \phi} + A_3 \frac{\partial z}{\partial \phi}. \tag{II.18.83}$$

From (II.18.78) and (II.18.81) we get

$$\bar{A}_1 = (xy)(\sin\theta\cos\phi) + (2y - z^2)(\sin\theta\sin\phi) + xz(\cos\theta)$$

or,

$$\begin{aligned} \bar{A}_1 = {} & (r^2 \sin^2\theta \cos\phi \sin\phi)\sin\theta\cos\phi + (2r\sin\theta\sin\phi - r^2\cos^2\theta)\sin\theta\sin\phi \\ & + (r^2\sin\theta\cos\phi\cos\theta)\cos\theta. \end{aligned}$$

Similarly from (II.18.82) and (II.18.83) we get

$$\begin{aligned} \bar{A}_2 = {} & (r^2\sin^2\theta\sin\phi\cos\phi)(r\cos\theta\cos\phi) + (2r\sin\theta\sin\phi - r^2\cos^2\theta) \\ & (r\cos\theta\sin\phi) + (r^2\sin\theta\cos\phi\cos\theta)(-r\sin\theta), \end{aligned}$$

$$\begin{aligned} \bar{A}_3 = {} & (r^2\sin^2\theta\sin\phi\cos\phi)(-r\sin\theta\sin\phi) + (2r\sin\theta\sin\phi - r^2\cos^2\theta) \\ & (r\sin\theta\cos\phi) + (r^2\sin\theta\cos\phi\cos\theta)(0). \end{aligned}$$

Example 34. Prove that the scalar product of a relative covariant vector of weight w and a relative contravariant vector of weight w' is a relative scalar of weight $w + w'$.

Let A^i be a relative contravariant vector of weight w and B_i be a relative covariant vector of weight w'. To prove that A^iB_i is a relative scalar of weight $w+w'$, we are to show that

$$\bar{A}^i\bar{B}_i = A^iB_i\left|\frac{\partial x}{\partial \bar{x}}\right|^{w+w'}$$

Now

$$\bar{A}^i = A^p\frac{\partial \bar{x}^i}{\partial x^p}\left|\frac{\partial x}{\partial \bar{x}}\right|^{w}$$

$$\bar{B}_i = B_q\frac{\partial x^q}{\partial \bar{x}^i}\left|\frac{\partial x}{\partial \bar{x}}\right|^{w'}.$$

Then,

$$\begin{aligned}\bar{A}^i\bar{B}_i &= A^pB_q\frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^i}\left|\frac{\partial x}{\partial \bar{x}}\right|^{w+w'}\\ &= A^pB_q\delta^q_p\left|\frac{\partial x}{\partial \bar{x}}\right|^{w+w'}\\ &= A^pB_p\left|\frac{\partial x}{\partial \bar{x}}\right|^{w+w'}\\ &= A^iB_i\left|\frac{\partial x}{\partial \bar{x}}\right|^{w+w'}.\end{aligned}$$

Example 35. If A^{ij} and A_{ij} are components of relative tensors of weight w, show that

$$|\bar{A}^{ij}| = |A^{ij}|\left|\frac{\partial x}{\partial \bar{x}}\right|^{w-2}$$

and

$$|\bar{A}_{ij}| = |A_{ij}|\left|\frac{\partial x}{\partial \bar{x}}\right|^{w+2}.$$

By the definition of a relative tensor, we have

$$\bar{A}^{ij} = A^{\alpha\beta}\frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial \bar{x}^j}{\partial x^\beta}\left|\frac{\partial x}{\partial \bar{x}}\right|^{w} \qquad \text{(II.18.84)}$$

and

$$\bar{A}_{ij} = A_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^i}\frac{\partial x^\beta}{\partial \bar{x}^j}\left|\frac{\partial x}{\partial \bar{x}}\right|^{w}. \qquad \text{(II.18.85)}$$

From (II.18.84) and (II.18.85) and by virtue of

$$\left|\frac{\partial \bar{x}^i}{\partial x^\alpha}\right| = \left|\frac{\partial \bar{x}^j}{\partial x^\beta}\right| = \left|\frac{\partial \bar{x}}{\partial x}\right|,$$

we get

$$|\bar{A}^{ij}| = |A^{\alpha\beta}| \left|\frac{\partial \bar{x}}{\partial x}\right| \left|\frac{\partial \bar{x}}{\partial x}\right| \left|\frac{\partial x}{\partial \bar{x}}\right|^{w}$$

and

$$|\bar{A}_{ij}| = |A_{\alpha\beta}| \left|\frac{\partial x}{\partial \bar{x}}\right| \left|\frac{\partial x}{\partial \bar{x}}\right| \left|\frac{\partial x}{\partial \bar{x}}\right|^{w}$$

But

$$\left|\frac{\partial \bar{x}}{\partial x}\right| = \left|\frac{\partial x}{\partial \bar{x}}\right|^{-1} \quad \text{and} \quad |A_{\alpha\beta}| = |\bar{A}_{ij}|.$$

Hence

$$|\bar{A}^{ij}| = |A^{ij}| \left|\frac{\partial x}{\partial \bar{x}}\right|^{w-2}$$

and

$$|\bar{A}_{ij}| = |A_{ij}| \left|\frac{\partial x}{\partial \bar{x}}\right|^{w+2}$$

Example 36. If A_i is a covariant vector, examine whether $\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$ are the components of a tensor.

Since A_i is a covariant vector, we have

$$\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p. \tag{II.18.86}$$

Differentiating both sides of (II.18.86) with respect to $\bar{x}^j$ we get

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial A_p}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j} \frac{\partial x^p}{\partial \bar{x}^i} + A_p \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i}. \tag{II.18.87}$$

Interchanging i and j we have

$$\frac{\partial \bar{A}_j}{\partial \bar{x}^i} = \frac{\partial A_p}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^j} + A_p \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}. \tag{II.18.88}$$

Subtracting (II.18.88) from (II.18.87) we get

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} - \frac{\partial \bar{A}_j}{\partial \bar{x}^i} = \frac{\partial A_p}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j} \frac{\partial x^p}{\partial \bar{x}^i} - \frac{\partial A_p}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^j}.$$

Replacing the dummy index p by k and k by p in the second term of the right hand side we get

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} - \frac{\partial \bar{A}_j}{\partial \bar{x}^i} = \left(\frac{\partial A_p}{\partial x^k} - \frac{\partial A_k}{\partial x^p}\right) \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^k}{\partial \bar{x}^j}.$$

That is,

$$\bar{B}_{ij} = B_{pk} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^k}{\partial \bar{x}^j}, \tag{II.18.89}$$

where

$$B_{pk} = \frac{\partial A_p}{\partial x^k} - \frac{\partial A_k}{\partial x^p}.$$

From (II.18.89), it follows that B_{ij} is a covariant tensor of order 2.

Example 37. If a_{ij} is a covariant tensor of order 2 and $|a_{ij}| = a$, then show that $\sqrt{a}$ is a relative tensor of order zero and weight 1.

From the transformation law, we have

$$\bar{a}_{ij} = a_{pq} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j}.$$

Taking determinants of both the sides we get

$$|\bar{a}_{ij}| = |a_{pq}| \left|\frac{\partial x^p}{\partial \bar{x}^i}\right| \left|\frac{\partial x^q}{\partial \bar{x}^j}\right|,$$

$$\text{or, } \bar{a} = a \left|\frac{\partial x}{\partial \bar{x}}\right|^2$$

Taking square root, we get

$$\sqrt{\bar{a}} = \sqrt{a} \left|\frac{\partial x}{\partial \bar{x}}\right|,$$

which shows that $\sqrt{a}$ is a relative tensor of order zero and weight 1.

Example 38. If $A_{ij} = 0$ for $i \neq j$ and $A_{ij} \neq 0$ for $i = j$, show that the conjugate tensor

$$B^{ij} = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad B^{ii} = \frac{1}{A_{ii}} \quad \text{(summation not implied)}.$$

Let

$$A = |A_{ij}|.$$

Since

$$A_{ij} = 0 \text{ for } i \neq j \text{ and } \quad A_{ii} \neq 0,$$

we have

$$|A_{ij}| = A = \begin{vmatrix} A_{11} & 0 & 0 & \dots & 0 \\ 0 & A_{22} & 0 & \dots & 0 \\ 0 & 0 & A_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_{nn} \end{vmatrix}$$

$$= A_{11} A_{22} \dots A_{nn}.$$

Since B^{ij} is the conjugate tensor of A_{ij}, we have

$$B^{ij} = \frac{\text{cofactor of } A_{ij} \text{ in } |A_{ij}|}{A}.$$

Evidently

$$B^{12} = \frac{\text{cofactor of } A_{12} \text{ in } |A_{ij}|}{A} = \frac{0}{A} = 0.$$

Generalizing this result, we get

$$B^{ij} = 0 \quad \text{for} \quad i \neq j.$$

Now

$$B^{11} = \frac{\text{cofactor of } A_{11} \text{ in } |A_{ij}|}{A} = \frac{A_{22}A_{33}\ldots A_{nn}}{A_{11}A_{22}\ldots A_{nn}} = \frac{1}{A_{11}}.$$

Similarly

$$B^{22} = \frac{1}{A_{22}}.$$

Hence in this way

$$B^{ii} = \frac{1}{A_{ii}}, \quad \text{for all } i$$

i.e.,

$$B^{ij} = \frac{1}{A_{ij}} \quad \text{for} \quad i = j.$$

II.19 Exercises

Exercise 1 . The components A^i of a contravariant vector in a coordinate system (x^i) are taken as follows :

$A^1 = f$, where f is an arbitrary function of the x^i's and $A^j = 0, \quad j = 2, \ldots, n.$

Show that the components $\bar{A}^i$ in any other system $(\bar{x}^i)$ are given by

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^1} f, \qquad i = 1, 2, \ldots, n.$$

Exercise 2 . If the relation $b^i_j v_i = 0$ holds for an arbitrary covariant vector v_i, show that $b^i_j = 0$.

Exercise 3 . If the relation $a_{ij} v^i v^j = 0$ holds for an arbitrary contravariant vector v^i, show that $a_{ij} + a_{ji} = 0$.

Exercise 4 . If $a_{ij}u^iu^j = b_{ij}u^iu^j$ for an arbitrary contravariant vector u^i, then show that $a_{ij} + a_{ji} = b_{ij} + b_{ji}$.

Exercise 5 . If the equality $a^i_j v_i = \sigma v_j$ holds for every covariant vector v_i, where σ is a scalar, show that $a^i_j = \sigma\delta^i_j$.

Exercise 6 . If a_{ij} is a skew-symmetric tensor, prove that

$$\left(\delta^i_j\delta^k_l + \delta^i_l\delta^k_j\right) a_{ik} = 0.$$

Exercise 7 . Prove that δ^i_j is a mixed tensor of type $(1, 1)$.

Exercise 8 . Show that in S_n a symmetric covariant tensor of second order has at most $\frac{1}{2}n(n+1)$ different components.

Exercise 9 . If A_{ijk} is completely symmetric and the indices run from 1 to n, show that the number of distinct components of A_{ijk} is $\frac{n(n+1)(n+2)}{6}$.

Exercise 10 . If a_{ij} $(\neq 0)$ are the components of a covariant tensor of order 2 such that $ba_{ij} + ca_{ji} = 0$, where b and c are non-zero scalars, show that either $b = c$ and a_{ij} is skew-symmetric or $b = -c$ and a_{ij} is symmetric.

Exercise 11 . Show that the determinant of a tensor of type $(1, 1)$ is an invariant.

Exercise 12 . If B_{ij} are components of a covariant tensor of second order and C^i, D^j are components of two contravariant vectors, show that $B_{ij}C^iD^j$ is an invariant.

Exercise 13 . If $a_{ij}u^iu^j$ is an invariant for an arbitrary contravariant vector u^i, show that $a_{ij} + a_{ji}$ is a tensor.

Exercise 14 . Show that there is no distinction between contravariant and covariant vectors when the transformations are of the type $\bar{x}^i = a^i_m x^m + d^i$, where d^i and a^i_m are constants such that $a^i_r a^i_m = \delta^r_m$ (i summed).

Exercise 15 . Prove that the transformations of contravariant vectors form a group.

Exercise 16 . If $A^i_{jk}B^{jk} = C^i$, where C^i is a contravariant vector and B^{jk} is an arbitrary symmetric tensor, show that $A^i_{jk} + A^i_{kj}$ is a tensor.

Exercise 17 . If the relation $A_{ijkl}u^iv^ju^kv^l = 0$ holds for arbitrary contravariant vectors u^i, v^i where A_{ijkl} satisfies the relations

$$A_{ijkl} + A_{ijlk} = 0, \qquad A_{ijkl} + A_{jikl} = 0 \quad \text{and} \quad A_{ijkl} + A_{iklj} + A_{iljk} = 0,$$

show that $A_{ijkl} = 0$.

Exercise 18 . If A_{ijkl} is a tensor satisfying the relations

$$A_{ijkl} + A_{ijlk} = 0, \qquad A_{ijkl} + A_{jikl} = 0 \quad \text{and} \quad A_{ijkl} + A_{iklj} + A_{iljk} = 0$$

and a_{lm} is a non-zero tensor such that

$$A_{hijk}a_{lm} + A_{lmhi}a_{jk} + A_{jklm}a_{hi} = 0,$$

prove that $A_{hijk} = 0$.

Exercise 19 . Prove that the transformation of tensors of type $(1, 1)$ form a group.

Exercise 20 . Show that in S_n $(n > 2)$ no relation of the form

$$\alpha a_{ij}b_{hk} + \beta a_{ih}b_{jk} + \gamma a_{ik}b_{hj} = 0$$

can exist, where α, β, γ are three scalars, a_{ij} is a symmetric tensor such that $|a_{ij}| \neq 0$ and b_{ij} is a skew-symmetric tensor.

Exercise 21 . If the relation $a_{ij}u^iu^j = 0$ holds for all vectors u^i such that $u^i\lambda_i = 0$, where λ_i is a given covariant vector, show that

$$a_{ij} + a_{ji} = \lambda_i v_j + \lambda_j v_i, \qquad v_j \text{ is some covariant vector.}$$

Exercise 22 . If the equality $a^i_j u_i = \sigma u_j$ holds for any covariant vector u_i such that $u_i v^i = 0$, where v^i is a given contravariant vector, show that $a^i_j = \sigma\delta^i_j + \lambda_j v^i$, where λ_i is some covariant vector.

Exercise 23 . If x, y, z are the components of a contravariant vector in rectangular Cartesian coordinates in a 3-dimensional space, show that the components of the vector in cylindrical polar coordinates r, θ, z are

$$x\cos\theta + y\sin\theta, \quad -\frac{x}{r}\sin\theta + \frac{y}{r}\cos\theta, \quad z.$$

Exercise 24 . If x, y, z are the components of a covariant vector in rectangular Cartesian coordinates in E_3, find the components of the vector in spherical coordinates r, θ, ϕ.

Exercise 25 . The components of a symmetric covariant tensor a_{ij} in a rectangular Cartesian coordinates in E_3 are given by

$$\begin{aligned} a_{ij} &= 1 \quad \text{for } i = j \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

and its components in
(i) cylindrical coordinates r, θ, z
and (ii) spherical coordinates r, θ, ϕ.

Exercise 26. If A^i_j is a mixed tensor, prove that $C_j = A^i_{ij}$ is a covariant vector.

Exercise 27. If $w = (2, 1, 1)$ is a contravariant vector in a coordinate system (x^1, x^2, x^3) in R^3, find its components in the coordinate system (x'^1, x'^2, x'^3) where

$$\begin{aligned} x'^1 &= 3x^1 - 3x^2 + 2x^3, \\ x'^2 &= 2x^2 + 3x^3, \\ x'^3 &= x^1 + x^2 + 2x^3. \end{aligned}$$

Exercise 28. If a^{ij} is a contravariant tensor such that $|a^{ij}| \neq 0$, show that $|a^{ij}|$ is a relative invariant of weight -2.

Exercise 29. If a_{ij} is a tensor such that $|a_{ij}| \neq 0$ and b^{ij} is the cofactor of a_{ij} in $|a_{ij}|$, examine whether b^{ij} is a relative tensor.

Exercise 30. If A_{ij} is skew-symmetric, then show that

$$\left(B^i_j B^k_p + B^i_p B^k_j\right) A_{ik} = 0.$$

Exercise 31. Evaluate $\left(\delta^i_j \delta^k_l + \delta^i_l \delta^k_j\right) a_{ik}$, where a_{ij} is a skew-symmetric tensor of order 2.

III Tensor Calculus

III.0 Introduction

In the previous chapter we became familiar with a type of mathematical object called 'tensor' and some algebraic operations defined on them. In this chapter we devote ourselves towards the study of 'tensor calculus'.

First let us consider a scalar function $\phi(x^1, x^2, \ldots, x^n)$ of n coordinates (x^i) in S_n. We have already seen (II.3) that $\frac{\partial \phi}{\partial x^i}$ form the components of a covariant vector, *i.e.*, the partial derivatives of a tensor of type $(0,0)$ is a tensor of type $(0,1)$.

Next, let us consider the components A_r of a covariant vector in (x^i) coordinate system. Let $\bar{A}_s$ denote its components in another coordinate system $(\bar{x}^i)$. Then we have

$$\bar{A}_s = \frac{\partial x^r}{\partial \bar{x}^s} A_r. \tag{III.0.1}$$

Differentiating both sides of the above equation with respect to $\bar{x}^k$, $(k = 1, 2, \ldots, n)$ we find

$$\begin{aligned} \frac{\partial \bar{A}_s}{\partial \bar{x}^k} &= \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^r}{\partial \bar{x}^s}\right) A_r + \frac{\partial x^r}{\partial \bar{x}^s}\frac{\partial A_r}{\partial \bar{x}^k} \\ &= \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^s} A_r + \frac{\partial x^r}{\partial \bar{x}^s}\frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial A_r}{\partial x^p}. \end{aligned} \tag{III.0.2}$$

The presence of the first term in the right hand side of (III.0.2) asserts that partial differentiation of a covariant vector does not yield a tensor.

Since the partial derivatives of a tensor are not always tensors, it is important to define a new type of differentiation of tensors in S_n which when applied to a tensor of any type always gives another tensor. But, this is not possible in S_n unless some additional structure is imposed in S_n.

In the three dimensional Euclidean space E_3 with rectangular Cartesian coordinate system the square of the distance ds between two neighbouring points $P(x, y, z)$ and $Q(x + dx, y + dy, z + dz)$ is given by

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{III.0.3}$$

In an n-dimensional Euclidean space E_n with rectangular Cartesian coordinate system we can express the square of the distance ds between two points $P(x^i)$ and $Q(x^i + dx^i)$ by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2. \tag{III.0.4}$$

The laws of transformation from three dimensional rectangular Cartesian coordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ to cylindrical coordinate system (x^1, x^2, x^3) is given by

$$\begin{aligned} \bar{x}^1 &= x^1 \cos x^2, \\ \bar{x}^2 &= x^1 \sin x^2, \\ \bar{x}^3 &= x^3 \end{aligned} \tag{III.0.5}$$

where $x^1 \geq 0, \quad 0 \leq x^2 \leq 2\pi$ and $-\infty < x^3 < \infty$.
In this case the square of the distance between the points $P(x^1, x^2, x^3)$ and $Q(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ is given by

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (dx^3)^2. \tag{III.0.6}$$

In case of spherical coordinate system we have the law of transformation as

$$\begin{aligned} \bar{x}^1 &= x^1 \sin x^2 \cos x^3, \\ \bar{x}^2 &= x^1 \sin x^2 \sin x^3, \\ \bar{x}^3 &= x^1 \cos x^2 \end{aligned} \tag{III.0.7}$$

where $x^1 \geq 0, \quad 0 \leq x^2 \leq \pi, \quad 0 \leq x^3 \leq 2\pi$. Here

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1 \sin x^2)^2 (dx^3)^2. \tag{III.0.8}$$

Let us consider any curvilinear coordinate system (u^i) in E_3. Let the position vectors of two neighbouring points P and Q be $\vec{r}$ and $\vec{r} + d\vec{r}$. Then

$$\begin{aligned} d\vec{r} &= \frac{\partial \vec{r}}{\partial u^1} du^1 + \frac{\partial \vec{r}}{\partial u^2} du^2 + \frac{\partial \vec{r}}{\partial u^3} du^3 \\ &= \vec{\alpha}_1 du^1 + \vec{\alpha}_2 du^2 + \vec{\alpha}_3 du^3, \qquad \text{say,} \end{aligned} \tag{III.0.9}$$

where $\vec{\alpha}_i = \frac{\partial \vec{r}}{\partial u^i}$ $(i = 1, 2, 3)$.
Now

$$\begin{aligned} ds^2 &= d\vec{r} \cdot d\vec{r} \\ &= \vec{\alpha}_1 \cdot \vec{\alpha}_1 (du^1)^2 + \vec{\alpha}_1 \cdot \vec{\alpha}_2 (du^1)(du^2) + \vec{\alpha}_1 \cdot \vec{\alpha}_3 (du^1)(du^3) \\ &\quad + \vec{\alpha}_2 \cdot \vec{\alpha}_1 (du^2)(du^1) + \vec{\alpha}_2 \cdot \vec{\alpha}_2 (du^2)^2 + \vec{\alpha}_2 \cdot \vec{\alpha}_3 (du^2)(du^3) \\ &\quad + \vec{\alpha}_3 \cdot \vec{\alpha}_1 (du^3)(du^1) + \vec{\alpha}_3 \cdot \vec{\alpha}_2 (du^3)(du^2) + \vec{\alpha}_3 \cdot \vec{\alpha}_3 (du^3)^2 \\ &= \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq} du^p du^q, \end{aligned} \tag{III.0.10}$$

where

$$g_{pq} = \vec{\alpha}_p \cdot \vec{\alpha}_q. \tag{III.0.11}$$

In special theory of relativity the square of the distance is formulated as

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2 (dx^4)^2 \tag{III.0.12}$$

where c denotes the speed of light in vacuum.

Thus we see that the expressions for square of the distance ds between two points are different for different coordinate systems.

On 10th June, 1854 in a lecture (under the heading "Über die Hypothesen welche der Geometrie zu Grunde Liegen") Bernhard Riemann (1826-1866) introduced the idea of a metric (which was later named after himself) which generalizes all the above ideas, by assuming that the distance between two neighbouring points is independent of coordinate system.

In the next section we like to discuss about the Riemannian metric and the Riemannian space.

III.1 Riemannian Space

III.1.1 Riemannian Metric

In S_n let us express the distance ds between two neighbouring points $P(x^i)$ and $Q(x^i + dx^i)$ by the quadratic differential formula

$$ds^2 = g_{ij}dx^i dx^j \qquad (i, j = 1, 2, \ldots, n) \tag{III.1.1}$$

in accordance with summation convention, where g_{ij} are arbitrary functions of the coordinates x^i, subject to the restriction

$$g = |g_{ij}| \neq 0$$

and ds is considered to be an invariant.

Then ds is called the line element and the quadratic form $g_{ij}dx^i dx^j$ is called the Riemannian metric. The quantities g_{ij} are called the coefficients of the Riemannian metric. Together they form the components of a rank two symmetric tensor (as will be shown below) which is called the metric tensor. The space S_n, endowed with such a structure is called a Riemannian space and is denoted by V_n.

In the examples cited in (III.0.3), (III.0.4), (III.0.6) and (III.0.8) we observe that ds^2 is non-negative. But in case of (III.0.12) it may be positive, negative or zero.

If ds^2 is negative then ds is a complex number, in that case ds is called an elementary interval. If ds is real it is called an elementary distance.

Note : Some authors prefers another definition so that ds^2 is always non-negative. They consider

$$ds^2 = \epsilon g_{ij}dx^i dx^j$$

where ϵ is equal to $+1$ or -1 such that ds^2 is always non-negative. This numerical factor ϵ is called the indicator.

Example 1. In case of the n dimensional Euclidean space E_n with rectangular Cartesian coordinates, the square of the line element is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2. \tag{III.1.2}$$

What is the metric tensor?

In this case

$$\begin{aligned} g_{ij} &= 1, && \text{if } i = j \\ &= 0, && \text{otherwise.} \end{aligned}$$

So, the Euclidean metric is a particular example of Riemannian metric.

Example 2. If

$$ds^2 = (dx)^2 + 2\cos\phi dxdy + (dy)^2 \tag{III.1.3}$$

then

$$g_{11} = 1 = g_{22} \quad \text{and} \quad g_{12} = \cos\phi = g_{21}.$$

Theorem 1.1 : The coefficients g_{ij} of the Riemannian metric is a covariant tensor of rank 2.

Proof : Since ds^2 is an invariant, it follows that $g_{ij}dx^i dx^j$ is an invariant. Again since dx^i is an arbitrary contravariant vector, applying the quotient law we conclude that g_{ij} are the components of a tensor of type $(0,2)$ or of rank 2.
This completes the proof.
Note : The tensor g_{ij} is called the fundamental tensor or metric tensor of V_n.

Theorem 1.2 : The fundamental metric tensor g_{ij} is symmetric.

Proof : Let us write

$$\begin{aligned} g_{ij} &= \frac{1}{2}(g_{ij} + g_{ji}) + \frac{1}{2}(g_{ij} - g_{ji}) \\ &= (A_{ij} + B_{ij}) \end{aligned} \tag{III.1.4}$$

where

$$A_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) \tag{III.1.5}$$

and

$$B_{ij} = \frac{1}{2}(g_{ij} - g_{ji}). \tag{III.1.6}$$

Now

$$\begin{aligned} A_{ji} &= \frac{1}{2}(g_{ji} + g_{ij}) \\ &= A_{ij} \end{aligned}$$

and

$$B_{ji} = \frac{1}{2}(g_{ji} - g_{ij})$$

$$= -\frac{1}{2}(g_{ij} - g_{ji})$$
$$= -B_{ij}.$$

Therefore A_{ij} are the components of a symmetric tensor whereas B_{ij} are the components of a skew-symmetric tensor.

Now

$$g_{ij}dx^i dx^j = A_{ij}dx^i dx^j + B_{ij}dx^i dx^j$$

or,

$$(g_{ij} - A_{ij})\,dx^i dx^j = B_{ij}dx^i dx^j. \tag{III.1.7}$$

Also

$$B_{ij}dx^i dx^j = B_{ji}dx^j dx^i \quad \text{(interchanging the dummy indices } i \text{ and } j\text{)}$$
$$= -B_{ij}dx^i dx^j \quad \text{(since } B_{ij} \text{ is skew-symmetric).}$$

This implies

$$B_{ij}dx^i dx^j = 0.$$

So

$$g_{ij}dx^i dx^j = A_{ij}dx^i dx^j.$$

Since A_{ij} is symmetric, we can take g_{ij} as a symmetric tensor without loss of generality.

Theorem 1.3 : The number of independent components of the fundamental metric tensor is at most $\frac{n(n+1)}{2}$.

Proof : Since g_{ij} is a covariant tensor of rank 2, it has the following n^2 components :

$$\begin{matrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & g_{nn}. \end{matrix}$$

Number of independent components along the principal diagonal of the above array (*i.e.*, number of terms with repeated indices) is n.

Number of components with unequal suffixes is $n^2 - n$. Since g_{ij} is symmetric by interchanging the indices we get the same component (*i.e.*, $g_{12} = g_{21}, g_{13} = g_{31}, g_{23} = g_{32}, \ldots, g_{n-1n} = g_{nn-1}$). Hence the number of independent components of g_{ij} with unequal suffixes reduces to $\frac{n^2-n}{2}$.

Therefore, the maximum number of independent components of g_{ij} is

$$n + \frac{n^2 - n}{2} = \frac{n^2 + n}{2}$$
$$= \frac{n(n+1)}{2}.$$

Example 3. Let us consider the three dimensional Euclidean space E_3. Find out the metric tensor in cylindrical polar coordinates.

The square of the line element ds in rectangular Cartesian coordinates is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{III.1.8}$$

If the coordinates of a point in rectangular Cartesian coordinates is (x^1, x^2, x^3) and in cylindrical coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, then we have

$$\begin{aligned} x^1 &= \bar{x}^1 \cos \bar{x}^2 \\ x^2 &= \bar{x}^1 \sin \bar{x}^2 \\ x^3 &= \bar{x}^3. \end{aligned} \tag{III.1.9}$$

From (III.1.8) we get

$$g_{11} = g_{22} = g_{33} = 1,$$
$$g_{12} = g_{21} = g_{23} = g_{32} = g_{13} = g_{31} = 0.$$

Now

$$\begin{aligned} \bar{g}_{11} &= g_{pq} \frac{\partial x^p}{\partial \bar{x}^1} \frac{\partial x^q}{\partial \bar{x}^1} \qquad (p, q = 1, 2, 3) \\ &= g_{11} \frac{\partial x^1}{\partial \bar{x}^1} \frac{\partial x^1}{\partial \bar{x}^1} + g_{22} \frac{\partial x^2}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^1} + g_{33} \frac{\partial x^3}{\partial \bar{x}^1} \frac{\partial x^3}{\partial \bar{x}^1} \\ &\quad \text{(since other } g_{ij}\text{'s vanish identically)} \\ &= \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 \\ &= \cos^2 \bar{x}^2 + \sin^2 \bar{x}^2 + 0 \\ &= 1. \end{aligned}$$

In a similar fashion

$$\begin{aligned} \bar{g}_{22} &= g_{pq} \frac{\partial x^p}{\partial \bar{x}^2} \frac{\partial x^q}{\partial \bar{x}^2} \\ &= g_{11} \frac{\partial x^1}{\partial \bar{x}^2} \frac{\partial x^1}{\partial \bar{x}^2} + g_{22} \frac{\partial x^2}{\partial \bar{x}^2} \frac{\partial x^2}{\partial \bar{x}^2} + g_{33} \frac{\partial x^3}{\partial \bar{x}^2} \frac{\partial x^3}{\partial \bar{x}^2} \\ &= \left(\frac{\partial x^1}{\partial \bar{x}^2}\right)^2 + \left(\frac{\partial x^2}{\partial \bar{x}^2}\right)^2 + \left(\frac{\partial x^3}{\partial \bar{x}^2}\right)^2 \\ &= (-\bar{x}^1 \sin \bar{x}^2)^2 + (\bar{x}^1 \cos \bar{x}^2)^2 + 0^2 \\ &= (\bar{x}^1)^2; \\ \bar{g}_{33} &= g_{pq} \frac{\partial x^p}{\partial \bar{x}^3} \frac{\partial x^q}{\partial \bar{x}^3} \\ &= g_{11} \frac{\partial x^1}{\partial \bar{x}^3} \frac{\partial x^1}{\partial \bar{x}^3} + g_{22} \frac{\partial x^2}{\partial \bar{x}^3} \frac{\partial x^2}{\partial \bar{x}^3} + g_{33} \frac{\partial x^3}{\partial \bar{x}^3} \frac{\partial x^3}{\partial \bar{x}^3} \end{aligned}$$

$$= \left(\frac{\partial x^1}{\partial \bar{x}^3}\right)^2 + \left(\frac{\partial x^2}{\partial \bar{x}^3}\right)^2 + \left(\frac{\partial x^3}{\partial \bar{x}^3}\right)^2$$
$$= 0^2 + 0^2 + 1$$
$$= 1;$$
$$\bar{g}_{12} = g_{pq}\frac{\partial x^p}{\partial \bar{x}^1}\frac{\partial x^q}{\partial \bar{x}^2}$$
$$= g_{11}\frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2} + g_{22}\frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2} + g_{33}\frac{\partial x^3}{\partial \bar{x}^1}\frac{\partial x^3}{\partial \bar{x}^2}$$
$$= \left(\cos \bar{x}^2\right)\left(-\bar{x}^1 \sin \bar{x}^2\right) + \left(\sin \bar{x}^2\right)\left(\bar{x}^1 \cos \bar{x}^2\right) + 0$$
$$= 0;$$
$$\bar{g}_{21} = g_{pq}\frac{\partial x^p}{\partial \bar{x}^2}\frac{\partial x^q}{\partial \bar{x}^1}$$
$$= 0$$

and so on.

Thus we get

$$\bar{g}_{11} = 1, \quad \bar{g}_{22} = \left(\bar{x}^1\right)^2, \quad \bar{g}_{33} = 1$$

and

$$\bar{g}_{pq} = 0 \qquad \text{if} \quad p \neq q.$$

Therefore

$$ds^2 = \bar{g}_{pq} d\bar{x}^p d\bar{x}^q$$
$$= \left(d\bar{x}^1\right)^2 + \left(\bar{x}^1\right)^2 \left(d\bar{x}^2\right)^2 + \left(d\bar{x}^3\right)^2. \qquad \text{(III.1.10)}$$

III.1.2 Reciprocal or Conjugate Tensor of the Fundamental Metric Tensor g_{ij}

We have already proved that (**Theorem 1.1**) g_{ij} represent a set of symmetric functions of the coordinates (x^i). Since $g = \det g_{ij} \neq 0$ for all points of the region, then by means of g_{ij} we can construct another set of n^2 functions as follows :

Let G^{ij} denote the cofactor of g_{ij} in $\det g_{ij}$. Let us consider the normalized cofactors

$$g(i,j) = \frac{G^{ij}}{g}. \qquad \text{(III.1.11)}$$

Since g_{ij} is symmetric, G^{ij} is also symmetric and hence from (III.1.11) it follows that $g(i,j)$ is also symmetric.

Now

$$g(i,j)\, g_{jk} = \frac{G^{ij}}{g} g_{jk}$$
$$= \frac{1}{g} g_{jk} (\text{cofactor of } g_{ij} \text{ in } \det g_{ij})$$
$$= 1, \qquad \text{if} \quad i = k,$$

$$= 0, \text{ otherwise.}$$

That is,

$$g(i,j)\, g_{jk} = \delta^i_k. \tag{III.1.12}$$

We have already shown (Art. II.5) that δ^i_k are the components of a mixed tensor of type $(1,1)$. Since g_{ij} is not an arbitrary tensor we can't apply quotient law to obtain that $g(i,j)$ is a tensor.

Let us consider an arbitrary contravariant vector A^i.

We define

$$B_j = g_{ij} A^i. \tag{III.1.13}$$

Since A^i is an arbitrary tensor of type $(1;0)$ and g_{ij} is a tensor of type $(0,2)$ it follows that B_j is an arbitrary tensor of type $(0,1)$. The existence of B_j is ensured by Cramer's rule.

Now

$$\begin{aligned} B_j\, g(k,j) &= g_{ij} A^i g(k,j) \\ &= g_{ij} g(k,j)\, A^i \\ &= \delta^k_i A^i \quad \text{[using (III.1.19)]} \\ &= A^k, \\ \text{i.e.,} \qquad B_j g(k,j) &= A^k. \end{aligned} \tag{III.1.14}$$

Since B_j is an arbitrary tensor of type $(0,1)$ and A^k is a tensor of type $(1,0)$ from (III.1.11) applying quotient law it follows that $g(k,j)$ are the components of a tensor of type $(2,0)$.

This tensor $g(k,j)$ is also symmetric. This is called fundamental contravariant tensor and is usually denoted by g^{kj}. Thus (III.1.12) can be rewritten as

$$g^{ij} g_{jk} = \delta^i_k. \tag{III.1.15}$$

Example 4. In case of E_3 with cylindrical coordinates we have seen (**Example 3**)

$$ds^2 = \bar{g}_{pq} d\bar{x}^p d\bar{x}^q$$

where

$$\bar{g}_{11} = 1, \quad \bar{g}_{22} = \left(\bar{x}^1\right)^2, \quad \bar{g}_{33} = 1 \quad \text{and} \quad \bar{g}_{pq} = 0 \quad \text{if} \quad p \neq q.$$

Find the conjugate tensor.

Then

$$\begin{aligned} \bar{g} &= \det \bar{g}_{pq} \\ &= \begin{vmatrix} \bar{g}_{11} & \bar{g}_{12} & \bar{g}_{13} \\ \bar{g}_{21} & \bar{g}_{22} & \bar{g}_{23} \\ \bar{g}_{31} & \bar{g}_{32} & \bar{g}_{33} \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & (\bar{x}^1)^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\bar{x}^1)^2.$$

Hence

$$\bar{g}^{11} = \frac{\text{cofactor of } \bar{g}_{11} \text{ in } \bar{g}}{\bar{g}} = \frac{(\bar{x}^1)^2}{(\bar{x}^1)^2} = 1,$$

$$\bar{g}^{22} = \frac{\text{cofactor of } \bar{g}_{22} \text{ in } \bar{g}}{\bar{g}} = \frac{1}{(\bar{x}^1)^2},$$

$$\bar{g}^{33} = \frac{\text{cofactor of } \bar{g}_{33} \text{ in } \bar{g}}{\bar{g}} = \frac{(\bar{x}^1)^2}{(\bar{x}^1)^2} = 1;$$

for $p \neq q$

$$\bar{g}^{pq} = \frac{\text{cofactor of } \bar{g}_{pq} \text{ in } \bar{g}}{\bar{g}} = 0.$$

Example 5. Find the conjugate metric tensor in a Riemannian space V_3 in which the distance ds is given by

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 3(dx^1)(dx^2) - 3(dx^2)(dx^1) + 2(dx^2)(dx^3) + 2(dx^3)(dx^2).$$

Comparing the above relation with

$$ds^2 = g_{ij} dx^i dx^j \qquad (i, j = 1, 2, 3)$$

we get

$$g_{11} = 5, \quad g_{22} = 3, \quad g_{33} = 4, \quad g_{12} = -3 = g_{21},$$
$$g_{23} = 2 = g_{32}, \quad g_{13} = 0 = g_{31}.$$

Therefore

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= 4.$$

Then

$$g^{11} = \frac{\text{cofactor of } g_{11} \text{ in } |g_{ij}|}{g} = \frac{1}{4}(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} = 2,$$

$$g^{22} = \frac{\text{cofactor of } g_{22} \text{ in } |g_{ij}|}{g} = \frac{1}{4}(-1)^{2+2}\begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = 5,$$

$$g^{33} = \frac{\text{cofactor of } g_{33} \text{ in } |g_{ij}|}{g} = \frac{1}{4}(-1)^{3+3}\begin{vmatrix} 5 & -3 \\ -3 & 3 \end{vmatrix} = \frac{3}{2},$$

$$g^{12} = \frac{\text{cofactor of } g_{12} \text{ in } |g_{ij}|}{g} = \frac{1}{4}(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 0 & 4 \end{vmatrix} = 3,$$

$$g^{23} = \frac{\text{cofactor of } g_{23} \text{ in } |g_{ij}|}{g} = \frac{1}{4}(-1)^{2+3}\begin{vmatrix} 5 & -3 \\ 0 & 2 \end{vmatrix} = -\frac{5}{2},$$

$$g^{13} = \frac{\text{cofactor of } g_{13} \text{ in } |g_{ij}|}{g} = \frac{1}{4}(-1)^{1+3}\begin{vmatrix} -3 & 0 \\ 3 & 2 \end{vmatrix} = -\frac{3}{2}.$$

In matrix form the conjugate metric tensor g^{ij} can be given by

$$\begin{pmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & 5 & -\frac{5}{2} \\ -\frac{3}{2} & -\frac{5}{2} & \frac{3}{2} \end{pmatrix}.$$

Note : Since $g_{ij}g^{jk} = \delta_i^k$, the product of the matrices corresponding to g_{ij} and g^{jk} will be an unit matrix of same order.

III.1.3 Associated Tensors, Lowering and Raising Indices

Let A^i be the components of a contravariant vector. From these we can define a covariant vector by setting

$$A_j = g_{ij}A^i. \tag{III.1.16}$$

Multiplying both the sides of (III.1.16) by g^{kj} we get

$$g^{kj}A_j = g^{kj}g_{ij}A^i = \delta_i^k A^i = A^k. \tag{III.1.17}$$

Hence we can write

$$A^k = g^{kj}A_j. \tag{III.1.18}$$

Similarly, from A^k we may obtain A_j.
The vectors A^i and A_i are called associates or associated vectors.

The operation of transforming A^i into A_i is named 'lowering the index' whereas the inverse operation, *i.e.*, transforming A_i into A^i is called 'raising the index'.

In S_n it was not possible to change the type of a tensor. But with the introduction of the Riemannian metric it is possible to change the type by using inner multiplication.

Now we show that like vectors the index of any tensor can be lowered or raised to form associate tensors.

From an arbitrary tensor A_{ij} of type $(0, 2)$ we can form the following tensors by raising indices

$$A_j^k = g^{ik} A_{ij} \qquad \text{or,} \qquad A_i^k = g^{kj} A_{ij}$$

and

$$A^{rs} = g^{ir} g^{js} A_{ij}.$$

These three tensors A_j^k, A_i^k and A^{rs} are called associates to the tensor A_{ij}. It can be easily seen that any of these tensors discussed above can be obtained from another by raising or lowering indices.

If we consider an Euclidean space E_n endowed with rectangular Cartesian coordinate system the metric tensor g_{ij} is nothing but the Kronecker delta δ_{ij}. So

$$A_i = \delta_{ij} A^j = A^i.$$

Hence in E_n with rectangular Cartesian coordinates the covariant vector A_i and the contravariant vector A^i need not be considered as different objects (otherwise we have to agree two tensors of different order may be equal), *i.e.*, in E_n with rectangular Cartesian coordinate system there is no distinction between covariant and contravariant vector.

For tensors of higher degree the situation is similar. That is why we consider the following pairs as different types of components of the same tensor of corresponding orders

$$(i) \quad A_i, \quad A^i;$$
$$(ii) \quad g_{ij}, \quad g^{ij}.$$

Note : We can define associate of any tensor, but reciprocal or conjugate can be defined only for a tensor of order 2.

Example 6. Prove that in a V_n

$$(g_{hj} g_{ik} - g_{hi} g_{jk})\, g^{hj} = (n-1)\, g_{ik},$$

where g_{ij} are the components of the fundamental metric tensor.

We have

$$\begin{aligned}
(g_{hj} g_{ik} - g_{hi} g_{jk})\, g^{hj} &= g_{hj} g_{ik} g^{hj} - g_{hi} g_{jk} g^{hj} \\
&= g_{hj} g^{hj} g_{ik} - g_{hi} \delta_k^h \\
&= \delta_j^j g_{ik} - g_{ki} \\
&= n g_{ik} - g_{ki} \quad (\text{since } \delta_j^j = \delta_1^1 + \delta_2^2 + \ldots + \delta_n^n \\
&\qquad\qquad = 1 + 1 + \ldots + 1 = n) \\
&= (n-1)\, g_{ik}.
\end{aligned}$$

Example 7. Let us consider a V_4 with line element

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2 (dx^4)^2.$$

Show how the components of contravariant and covariant vectors are related in this space.

Here

$$g_{11} = -1, \quad g_{22} = -1, \quad g_{33} = -1, \quad g_{44} = c^2 \quad \text{and} \quad g_{ij} = 0 \quad \text{for} \quad i \neq j.$$

Let us consider the components of a contravariant vector A^i.
Then we set

$$A_j = g_{ij} A^i. \tag{III.1.19}$$

We get

$$\begin{aligned} A_1 &= g_{i1} A^i &&= -A^1, \\ A_2 &= g_{i2} A^i &&= -A^2, \\ A_3 &= g_{i3} A^i &&= -A^3, \\ A_4 &= g_{i4} A^i &&= +c^2 A^4. \end{aligned} \tag{III.1.20}$$

Now

$$\begin{aligned} g &= \det g_{ij} \\ &= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & c^2 \end{vmatrix} \\ &= -c^2. \end{aligned}$$

Then

$$g^{11} = \frac{\text{cofactor of } g_{11} \text{ in } \det g_{ij}}{g} = \frac{c^2}{-c^2} = -1,$$

$$g^{22} = -1, \quad g^{33} = -1 \quad \text{and} \quad g^{44} = \frac{1}{c^2}.$$

We observe

$$\begin{aligned} A^1 &= g^{1j} A_j &&= g^{11} A_1 &&= -A_1, \\ A^2 &= g^{2j} A_j &&= g^{22} A_2 &&= -A_2, \\ A^3 &= g^{3j} A_j &&= g^{33} A_3 &&= -A_3, \\ A^4 &= g^{4j} A_j &&= g^{44} A_4 &&= \tfrac{1}{c^2} A_4. \end{aligned} \tag{III.1.21}$$

This shows the relations given by (III.1.20) and (III.1.21) are identical. Hence A_i and A^i are associates.

Example 8. If ϕ is an invariant, prove that

$$\frac{\partial \phi}{\partial x^l} (g_{hj} g_{ik} - g_{hi} g_{jk}) g^{hl} = \frac{\partial \phi}{\partial x^j} g_{ik} - \frac{\partial \phi}{\partial x^i} g_{jk}.$$

Since ϕ is an invariant, $\frac{\partial \phi}{\partial x^i}$ are the components of a covariant vector. Let us write

$$\phi_i = \frac{\partial \phi}{\partial x^i}.$$

Then

$$\begin{aligned}\frac{\partial \phi}{\partial x^l}(g_{hj}g_{ik} - g_{hi}g_{jk})g^{hl} &= \phi_l(g_{hj}g_{ik}g^{hl} - g_{hi}g_{jk}g^{hl}) \\ &= \phi_l(g_{ik}g_{hj}g^{hl} - g_{jk}g_{hi}g^{hl}) \\ &= \phi_l(g_{ik}\delta^l_j - g_{jk}\delta^l_i) \\ &= g_{ik}\phi_j - g_{jk}\phi_i \\ &= g_{ik}\frac{\partial \phi}{\partial x^j} - g_{jk}\frac{\partial \phi}{\partial x^i}.\end{aligned}$$

Example 9. Prove that

$$g^{ij}\frac{\partial g_{ij}}{\partial x^k} + g_{ij}\frac{\partial g^{ij}}{\partial x^k} = 0.$$

We know that

$$g_{ij}g^{ij} = \delta^i_j = \delta^1_1 + \delta^2_2 + \ldots + \delta^n_n = n.$$

Differentiating with respect to x^k we get

$$\frac{\partial g_{ij}}{\partial x^k}g^{ij} + g_{ij}\frac{\partial g^{ij}}{\partial x^k} = 0.$$

Sometimes there is a confusion whether the idea of metric tensors g_{ij} is similar to the idea of metric axioms of a metric space. A metric tensor is of course not identical to the metric axioms of a metric space, but from it we can define a metric d that satisfies all the metric axioms, as shown in the following :

Let us consider any two distinct points P and Q of V_n. Let C be any curve defined by $x = x(t)$ where

$$x : [a, b] \to V_n, \quad t \in [a, b] \subset \mathcal{R}$$

and $x(a) = P$, $x(b) = Q$.

Then the arc length of the curve C is given by

$$L(C) = \int_a^b \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt.$$

We define $d(P, Q) = \inf L(C)$, for all curves joining P and Q. It can be easily seen that this function d satisfies all the metric axioms, i.e.,

$$d(P, P) = 0,$$

$$\begin{aligned} d(P,Q) &= d(Q,P), \\ d(P,Q)+d(Q,R) &\geq d(P,R) \end{aligned}$$

for any three points P, Q and R in V_n.

III.1.4 Magnitude or Length of a Vector

The square of the magnitude of a covariant vector A_i is defined by $g^{ij}A_iA_j$.
The square of the magnitude of a contravariant vector A^j is defined by $g_{ij}A^iA^j$.
Since

$$g_{ij}A^iA^j = A_jA^j = g^{ij}A_iA_j,$$

it follows that the associate vectors A_i and A^i are of the same length.

We shall use the notation $|A|$ to denote the magnitude of the vectors A_i and A^i.

Theorem 1.4 : The magnitude of a vector is an invariant.

Proof : Let us consider a vector whose covariant and contravariant components in (x^i) coordinate system are A_i and A^i respectively.

Let in $(\bar{x}^r)$ coordinate system the corresponding components be $\bar{A}_r$ and $\bar{A}^r$ respectively.

Then

$$\bar{A}^r = \frac{\partial \bar{x}^r}{\partial x^i}A^i \tag{III.1.22}$$

and

$$\bar{A}_r = \frac{\partial x^j}{\partial \bar{x}^r}A_j. \tag{III.1.23}$$

Now

$$\begin{aligned} \bar{A}^r\bar{A}_r &= \frac{\partial \bar{x}^r}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^r}A^iA_j \\ &= \delta_i^jA^iA_j \\ &= A^jA_j. \end{aligned}$$

III.1.5 Unit Vector

A covariant vector A_i is said to be a unit vector or a covariant vector of unit length if

$$g^{ij}A_iA_j = 1. \tag{III.1.24}$$

A contravariant vector A^i is said to be a unit vector if

$$g_{ij}A^iA^j = 1. \tag{III.1.25}$$

Example 10. In V_4 with line element

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2\left(dx^4\right)^2$$

show that the vector $\left(\sqrt{2}, 0, 0, \frac{\sqrt{3}}{c}\right)$ is a unit vector.

We find here $g_{11} = -1$, $g_{22} = -1$, $g_{33} = -1$, $g_{44} = c^2$, all other g_{ij}'s vanish identically.

Let A^i denote the components of the vector $\left(\sqrt{2}, 0, 0, \frac{\sqrt{3}}{c}\right)$.

Then

$$A^1 = \sqrt{2}, \quad A^2 = 0, \quad A^3 = 0, \quad A^4 = \frac{\sqrt{3}}{c}$$

So

$$\begin{aligned} g_{ij}A^iA^j &= g_{11}A^1A^1 + g_{22}A^2A^2 + g_{33}A^3A^3 + g_{44}A^4A^4 \\ &= (-1)\left(\sqrt{2}\right)^2 + (-1)(0)^2 + (-1)(0)^2 + c^2\left(\frac{\sqrt{3}}{c}\right)^2 \\ &= -2 + 3 \\ &= 1 \end{aligned}$$

Therefore A^i is a unit vector.

Note : The V_4 with the above metric is called Minkowski space-time of special theory of Relativity.

III.1.6 Null Vector

A contravariant vector A^i is said to be a null vector if

$$g_{ij}A^iA^j = 0. \tag{III.1.26}$$

A covariant vector A_i is said to be a null vector if

$$g^{ij}A_iA_j = 0. \tag{III.1.27}$$

Since the components of a zero vector are all zero it follows that zero vectors are null vectors. But the converse is not necessarily true. This can be understood from the following example :

Example 11. Let us consider a V_4 with the line element given by

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2\left(dx^4\right)^2.$$

Then the vector $\left(-1, -1, 1, \frac{\sqrt{3}}{c}\right)$ is a null vector in this space.

Here

$$g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = c^2$$

and

$$g_{ij} = 0 \quad \text{for} \quad i \neq j.$$

Let us consider the vector $\left(-1, -1, 1, \frac{\sqrt{3}}{c}\right)$.

Then

$$\begin{aligned} g_{ij}A^iA^j &= g_{11}A^1A^1 + g_{22}A^2A^2 + g_{33}A^3A^3 + g_{44}A^4A^4 \\ &= -1(-1)^2 - 1(-1)^2 - 1(-1)^2 + c^2\left(\frac{\sqrt{3}}{c}\right)^2 \\ &= 0. \end{aligned}$$

Hence the vector $\left(-1, -1, 1, \frac{\sqrt{3}}{c}\right)$ is a null vector in the above space.

III.1.7 Angle Between Two Non-null Vectors

Angle between two vectors is defined if and only if both of them are non-null. The angle θ between two non-null contravariant vectors A^i and B^i is defined by

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}}. \tag{III.1.28}$$

The angle θ between two non-null covariant vectors A_i and B_i is defined by

$$\cos\theta = \frac{g^{ij}A_iB_j}{\sqrt{g^{ij}A_iA_j}\sqrt{g^{ij}B_iB_j}}. \tag{III.1.29}$$

If A and B are two non-null vectors with respective covariant and contravariant components A_i, A^i and B_i, B^i, then the angle between them is given by

$$\cos\theta = \frac{A^iB_i}{\sqrt{A^iA_i}\sqrt{B^iB_i}} = \frac{A_iB^i}{\sqrt{A^iA_i}\sqrt{B^iB_i}}. \tag{III.1.30}$$

Note : The angle between two covariant vectors or two contravariant vectors is real if the Riemannian metric is positive definite.
Note : From (III.1.30) it follows that the angle between two vectors is an invariant.

III.1.8 Orthogonal Vectors

Two contravariant vectors A^i and B^j are said to be orthogonal if

$$g_{ij}A^iB^j = 0. \tag{III.1.31}$$

Two covariant vectors A_i and B_j are said to be orthogonal if

$$g^{ij}A_iB_j = 0. \tag{III.1.32}$$

From the definition it is clear that the angle between two orthogonal vectors is a right angle.

Also, it is obvious that the null vector is orthogonal with any vector.

Example 12. If $U^i = A^i + B^i$, where A^i and B^i are two orthogonal unit vectors, show that the square of the length of the vector U^i is 2.

From the given conditions we have

$$g_{ij}A^iA^j = 1,$$
$$g_{ij}B^iB^j = 1,$$
and
$$g_{ij}A^iB^j = 0.$$

Now, square of the length of the vector U^i

$$\begin{aligned} &= g_{ij}U^iU^j \\ &= g_{ij}\left(A^i + B^i\right)\left(A^j + B^j\right) \\ &= g_{ij}A^iA^j + g_{ij}A^iB^j + g_{ij}B^iA^j + g_{ij}B^iB^j \\ &= 1+0+0+1 \\ &= 2. \end{aligned}$$

Example 13. If A^i and B^i are orthogonal vectors each of length l, then prove that

$$\left(g_{hj}g_{ki} - g_{hk}g_{ji}\right)A^hB^jA^kB^i = -l^4.$$

Since A^i and B^i are orthogonal vectors each of length l we have,

$$g_{ij}A^iA^j = l^2,$$
$$g_{ij}B^iB^j = l^2$$
and
$$g_{ij}A^iB^j = 0.$$

Now

$$\begin{aligned} \left(g_{hj}g_{ki} - g_{hk}g_{ji}\right)A^hB^jA^kB^i &= g_{hj}g_{ki}A^hB^jA^kB^i - g_{hk}g_{ji}A^hB^jA^kB^i \\ &= \left(g_{hj}A^hB^j\right)\left(g_{ki}A^kB^i\right) - \left(g_{hk}A^hA^k\right)\left(g_{ji}B^jB^i\right) \\ &= 0 \times 0 - l^2 \times l^2 \\ &= -l^4. \end{aligned}$$

Example 14. If X_{ij} are the components of a symmetric covariant tensor and A^i, B^j are the components of two unit vectors both orthogonal to another vector C^k satisfying the conditions

$$X_{ij}A^i - \alpha g_{ij}A^i + \beta g_{ij}C^i = 0$$

and $$X_{ij}B^i - \alpha' g_{ij}B^i + \beta' g_{ij}C^i = 0$$

where $\alpha \neq \alpha'$, prove that A^i and B^i are orthogonal and $X_{ij}A^iB^j = 0$.

Since A^i and B^j are unit vectors,

$$\begin{aligned} g_{ij}A^iA^j &= 1, \\ g_{ij}B^iB^j &= 1. \end{aligned} \qquad \text{(III.1.33)}$$

Since A^i and B^j both are orthogonal to C^k, we have

$$\begin{aligned} g_{ij}A^iC^j &= 0, \\ g_{ij}B^iC^j &= 0. \end{aligned} \qquad \text{(III.1.34)}$$

Multiplying both sides of the relation

$$X_{ij}A^i - \alpha g_{ij}A^i + \beta g_{ij}C^i = 0$$

by B^j we get

$$\begin{aligned} X_{ij}A^iB^j - \alpha g_{ij}A^iB^j + \beta g_{ij}C^iB^j &= 0 \\ \text{or,} \quad X_{ij}A^iB^j = \alpha g_{ij}A^iB^j. & \end{aligned} \qquad \text{(III.1.35)}$$

Similarly multiplying both sides of the relation

$$X_{ij}B^i - \alpha' g_{ij}B^i + \beta' g_{ij}C^i = 0$$

by A^j we get

$$\begin{aligned} X_{ij}B^iA^j - \alpha' g_{ij}B^iA^j + \beta' g_{ij}C^iA^j &= 0 \\ \text{or,} \quad X_{ij}A^jB^i = \alpha' g_{ij}A^jB^i. & \end{aligned}$$

Interchanging dummy indices i and j we get from the above equation

$$X_{ji}A^iB^j = \alpha' g_{ij}A^iB^j.$$

This implies

$$X_{ij}A^iB^j = \alpha' g_{ij}A^iB^j. \qquad \text{(III.1.36)}$$

Combining (III.1.32) and (III.1.33)

$$\begin{aligned} \alpha g_{ij}A^iB^j &= \alpha' g_{ij}A^iB^j \\ i.e., \qquad (\alpha - \alpha')\, g_{ij}A^iB^j &= 0. \end{aligned}$$

This implies

$$g_{ij}A^iB^j = 0 \qquad (\text{since} \quad \alpha \neq \alpha') \qquad \text{(III.1.37)}$$

i.e., the vectors A^i and B^j are orthogonal.

Combining (III.1.32) and (III.1.34) we get

$$X_{ij}A^iB^j = 0.$$

Example 15. If a_{ij} are the components of a symmetric tensor satisfying

$$g_{ij}a_{hk} - g_{ih}a_{jk} + g_{jk}a_{hi} - g_{kh}a_{ij} = 0$$

show that $a_{ij} = \rho g_{ij}$, where ρ is a scalar.

Multiplying both sides of the given equation by g^{ij} we get

$$g^{ij}g_{ij}a_{hk} - g^{ij}g_{ih}a_{jk} + g^{ij}g_{jk}a_{hi} - g^{ij}g_{kh}a_{ij} = 0$$

or,

$$\delta^i_i a_{hk} - \delta^j_h a_{jk} + \delta^i_k a_{hi} - g_{kh}g^{ij}a_{ij} = 0$$

or,

$$na_{hk} - a_{hk} + a_{hk} - g_{kh}a = 0$$

(since $\delta^i_i = \delta^1_1 + \delta^2_2 + \ldots + \delta^n_n = 1 + 1 + \ldots + 1 = n$ and $g^{ij}a_{ij} = a$ is a scalar)
or,

$$na_{hk} = ag_{hk}$$

i.e.,

$$a_{hk} = \rho g_{hk}$$

where $\rho = \frac{a}{n}$ is a scalar.

Example 16. If $g_{ij} = 0$ for $i \neq j$ prove that $g^{ii} = \frac{1}{g_{ii}}$ (no summation over i).

Since $g_{ij} = 0$ for $i \neq j$, it follows that

$$\det g_{ij} = \begin{vmatrix} g_{11} & 0 & 0 & \ldots & 0 \\ 0 & g_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & g_{nn} \end{vmatrix}$$
$$= g_{11}g_{22}\cdots g_{nn}.$$

Now

$$g^{ii} = \frac{\text{cofactor of } g_{ii} \text{ in } \det g_{ij}}{\det g_{ij}}$$
$$= \frac{g_{11}g_{22}\cdots g_{i-1i-1}g_{i+1i+1}\cdots g_{nn}}{g_{11}g_{22}\cdots g_{nn}}$$
$$= \frac{1}{g_{ii}}.$$

Exercises

Exercise 1 . If g_{ij} denotes the fundamental metric tensor and g^{ij} denotes its reciprocal tensor prove that

$$g^{in}g_{nj} = \delta^i_j.$$

Exercise 2 . Find the conjugate metric tensor in a Riemannian metric space V_2 in which the distance ds is given by

$$ds^2 = (dx^1)^2 + 2\cos w\,(dx^1)(dx^2) + (dx^2)^2.$$

Exercise 3 . Show that in V_4 with the line element

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2(dx^4)^2,$$

the vector $\left(-1, 0, 0, \frac{1}{c}\right)$ is a null vector.

Exercise 4 . Let g_{ij} and g^{ij} be reciprocal symmetric tensors, u_i and v_i are components of two covariant vectors. If u^i and v^i are defined by

$$u^i = g^{ij}u_j, \qquad v^i = g^{ij}v_j$$

prove that

$$(i) \quad u_i = g_{ij}u^j,$$
$$(ii) \quad u^iv_i = u_iv^i.$$

Exercise 5 . If A^i and B^j are the components of orthogonal unit vectors, prove that

$$(g_{hj}g_{ik} - g_{hk}g_{ij})\,A^hB^iA^jB^k = 1.$$

III.2 Christoffel Symbols and their Properties

With the motivation to build up expressions involving the derivatives of a tensor which are again components of a tensor in 1869 E.B.Christoffel ["Über die transformation der homogenel differentialausdrücke zweiten Gradef", Journal für Math, 70(1869)] introduced two functions formed by certain combinations of the partial derivatives of the fundamental tensors g_{ij}, which are defined as follows :

III.2.1 *Christoffel Symbols of First Kind*

The Christoffel 3 index symbols of first kind are denoted by $[ij, k]$ and defined by

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right), \qquad (i,j,k = 1,2,\ldots,n). \qquad \text{(III.2.1)}$$

III.2.2 Christoffel Symbols of Second Kind

The Christoffel 3 index symbols of second kind are denoted by $\left\{ {l \atop i\,j} \right\}$ and defined by

$$\left\{ {l \atop i\,j} \right\} = g^{lk}\,[i\,j,k]$$
$$= \frac{1}{2} g^{lk} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right), \quad (i,j,k = 1,2,\ldots,n). \tag{III.2.2}$$

From (III.2.1) and (III.2.2) it follows that $[ij,k]$ and $\left\{ {l \atop i\,j} \right\}$ are mathematical objects formed by means of n^3 functions of coordinates (x^i).

Note : Different authors use different notations for Christoffel 3-index symbols, e.g. $\left[{i\,j \atop k} \right]$, $\Gamma_{ij,k}$ for $[i\,j,k]$ and $\left\{ {i\,j \atop k} \right\}$ and Γ^k_{ij} for $\left\{ {k \atop i\,j} \right\}$. The notations $\Gamma_{ij,k}$ and Γ^k_{ij} suggest that Christoffel symbols are tensors, but later we shall prove that they are not in general true. The present forms $[i\,j,k]$ and $\left\{ {k \atop i\,j} \right\}$ are most widely accepted, due to their familiarity with Einstein's summation convention.

III.2.3 Properties of Christoffel Symbols

1. Christoffel symbol of first kind is symmetric in first two indices, *i.e.*,

$$[i\,j,k] = [j\,i,k]$$

By definition

$$[j\,i,k] = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k} \right)$$
$$= \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right), \quad \text{since } g_{ij} \text{ is symmetric}$$
$$= [i\,j,k]\,. \tag{III.2.3}$$

2. Christoffel symbol of second kind is symmetric in second two indices, *i.e.*,

$$\left\{ {l \atop i\,j} \right\} = \left\{ {l \atop j\,i} \right\}.$$

We have

$$\left\{ {l \atop j\,i} \right\} = g^{kl}\,[j\,i,k]$$
$$= g^{kl}\,[i\,j,k], \quad \text{by property 1}$$
$$= \left\{ {l \atop i\,j} \right\}. \tag{III.2.4}$$

3. $[ij,k]+[kj,i]=\dfrac{\partial g_{ik}}{\partial x^j}$.

We have

$$\begin{aligned}[ij,k]+[kj,i] &= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)+\frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{kj}}{\partial x^i}\right)\\ &= \frac{1}{2}\times 2\frac{\partial g_{ik}}{\partial x^j}\\ &= \frac{\partial g_{ik}}{\partial x^j}. \qquad \text{(III.2.5)}\end{aligned}$$

4. $[ij,h]=g_{kh}\begin{Bmatrix} k \\ i\ j \end{Bmatrix}$.

By definition

$$\begin{Bmatrix} k \\ i\ j \end{Bmatrix}=g^{km}[ij,m].$$

Multiplying both sides of the above equation by g_{kh} we get

$$\begin{aligned}g_{kh}\begin{Bmatrix} k \\ i\ j \end{Bmatrix} &= g_{kh}g^{km}[ij,m]\\ &= \delta_h^m[ij,m]\\ &= [ij,h]. \qquad \text{(III.2.6)}\end{aligned}$$

5. $\dfrac{\partial g_{ij}}{\partial x^k}=g_{lj}\begin{Bmatrix} l \\ i\ k \end{Bmatrix}+g_{li}\begin{Bmatrix} l \\ j\ k \end{Bmatrix}$.

We have already proved

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x^k} &= [ik,j]+[jk,i]\\ &= g_{lj}\begin{Bmatrix} l \\ i\ k \end{Bmatrix}+g_{li}\begin{Bmatrix} l \\ j\ k \end{Bmatrix}, \quad \text{by (III.2.6).} \qquad \text{(III.2.7)}\end{aligned}$$

6. $\dfrac{\partial g^{im}}{\partial x^l}=-g^{ij}\begin{Bmatrix} m \\ l\ j \end{Bmatrix}-g^{km}\begin{Bmatrix} i \\ k\ l \end{Bmatrix}$.

We know that

$$g^{ij}g_{kj}=\delta_k^i.$$

Differentiating both the sides partially with respect to x^l we get

$$\frac{\partial g^{ij}}{\partial x^l}g_{kj}+g^{ij}\frac{\partial g_{kj}}{\partial x^l}=0$$

or,

$$\frac{\partial g^{ij}}{\partial x^l}g_{kj}=-g^{ij}\frac{\partial g_{kj}}{\partial x^l}.$$

Multiplying both the sides by g^{km}

$$g^{km}g_{kj}\frac{\partial g^{ij}}{\partial x^l} = -g^{ij}g^{km}\frac{\partial g_{kj}}{\partial x^l}$$

or,

$$\delta_j^m \frac{\partial g^{ij}}{\partial x^l} = -g^{ij}g^{km}\frac{\partial g_{kj}}{\partial x^l},$$

or,

$$\begin{aligned}\frac{\partial g^{im}}{\partial x^l} &= -g^{ij}g^{km}\frac{\partial g_{kj}}{\partial x^l} \qquad \text{(III.2.8)}\\ &= -g^{ij}g^{km}([k\,l,j]+[j\,l,k])\\ &= -g^{km}g^{ij}[k\,l,j]-g^{ij}g^{km}[j\,l,k]\\ &= -g^{km}\begin{Bmatrix} i \\ k\,l\end{Bmatrix} - g^{ij}\begin{Bmatrix} m \\ j\,l\end{Bmatrix}. \qquad \text{(III.2.9)}\end{aligned}$$

7. $\dfrac{\partial g_{ij}}{\partial x^k} - \dfrac{\partial g_{jk}}{\partial x^i} = [k\,j,i] - [i\,j,k]$.

We have

$$\begin{aligned}[k\,j,i]-[i\,j,k] &= \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{kj}}{\partial x^i}\right)-\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)\\ &= \frac{\partial g_{ij}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^i}. \qquad \text{(III.2.10)}\end{aligned}$$

8. If $g = |g_{ij}| > 0$, then $\begin{Bmatrix} i \\ i\,l\end{Bmatrix} = \dfrac{\partial \log\sqrt{g}}{\partial x^l}$.

We have

$$g = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{vmatrix}. \qquad \text{(III.2.11)}$$

Let G_{ht} denote the cofactor of g_{ht} in $|g_{ij}|$.

Then

$$g^{ht} = \frac{G_{ht}}{g},$$

i.e.,

$$G_{ht} = gg^{ht}. \qquad \text{(III.2.12)}$$

Differentiating both sides of the above equation (III.2.11) with respect to x^l, $(l = 1, 2, \ldots, n)$ we get

$$\frac{\partial g}{\partial x^l} = \begin{vmatrix} \frac{\partial g_{11}}{\partial x^l} & \frac{\partial g_{12}}{\partial x^l} & \cdots & \frac{\partial g_{1n}}{\partial x^l} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{vmatrix} + \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ \frac{\partial g_{21}}{\partial x^l} & \frac{\partial g_{22}}{\partial x^l} & \cdots & \frac{\partial g_{2n}}{\partial x^l} \\ g_{31} & g_{32} & \cdots & g_{3n} \\ \vdots & \vdots & \cdots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{vmatrix}$$

$$+\ldots+\begin{vmatrix} g_{11} & g_{12} & \ldots & g_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ g_{n-11} & g_{n-12} & \ldots & g_{n-1n} \\ \frac{\partial g_{n1}}{\partial x^l} & \frac{\partial g_{n2}}{\partial x^l} & \ldots & \frac{\partial g_{nn}}{\partial x^l} \end{vmatrix}$$

$$= \left(\frac{\partial g_{11}}{\partial x^l}G_{11} + \frac{\partial g_{12}}{\partial x^l}G_{12} + \ldots + \frac{\partial g_{1n}}{\partial x^l}G_{1n}\right)$$

$$+\left(\frac{\partial g_{21}}{\partial x^l}G_{21} + \frac{\partial g_{22}}{\partial x^l}G_{22} + \ldots + \frac{\partial g_{2n}}{\partial x^l}G_{2n}\right)$$

$$+\ldots+\left(\frac{\partial g_{n1}}{\partial x^l}G_{n1} + \frac{\partial g_{n2}}{\partial x^l}G_{n2} + \ldots + \frac{\partial g_{nn}}{\partial x^l}G_{nn}\right)$$

$$= \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^l}G_{ij}$$

$$= \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^l}gg^{ij}, \qquad \text{applying (III.2.12)}$$

$$= gg^{ij}\frac{\partial g_{ij}}{\partial x^l}, \qquad \text{summing for } i \text{ and } j$$

$$= gg^{ij}\left([i\,l,j] + [j\,l,i]\right), \qquad \text{using (III.2.5)}$$

$$= gg^{ij}\,[i\,l,j] + gg^{ij}\,[j\,l,i]$$

$$= g\left\{\begin{matrix} i \\ i\;l \end{matrix}\right\} + g\left\{\begin{matrix} j \\ j\;l \end{matrix}\right\}$$

$$= 2g\left\{\begin{matrix} i \\ i\;l \end{matrix}\right\}, \qquad \text{replacing the dummy index } j \text{ by } i.$$

Therefore

$$\left\{\begin{matrix} i \\ i\;l \end{matrix}\right\} = \frac{1}{2g}\frac{\partial g}{\partial x^l} = \frac{\partial}{\partial x^l}\left(\frac{1}{2}\log g\right) = \frac{\partial}{\partial x^l}\left(\log\sqrt{g}\right). \qquad \text{(III.2.13)}$$

Note : In case $g < 0$, in a similar manner we can deduce

$$\left\{\begin{matrix} i \\ i\;l \end{matrix}\right\} = \frac{\partial}{\partial x^l}\left(\log\sqrt{-g}\right).$$

Note : Since g is not an invariant, from (III.2.13) it does not follow that $\left\{\begin{matrix} i \\ i\;l \end{matrix}\right\}$ is a covariant vector.

Example 17. Calculate the Christoffel symbols $[i\,j,k]$ and $\left\{\begin{matrix} i \\ j\;k \end{matrix}\right\}$ corresponding to the metric

$$ds^2 = \left(dx^1\right)^2 + \left(x^1\right)^2\left(dx^2\right)^2 + \left(x^1\sin x^2\right)^2\left(dx^3\right)^2.$$

Comparing the expression

$$ds^2 = g_{ij}dx^i dx^j \tag{III.2.14}$$

with

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1 \sin x^2)^2 (dx^3)^2, \tag{III.2.15}$$

we find

$$\left.\begin{array}{lll} g_{11} = 1, & g_{22} = (x^1)^2, & g_{33} = (x^1 \sin x^2)^2 \\ \text{and } g_{ij} = 0 & \text{for } i \neq j & (i, j = 1, 2, 3). \end{array}\right\} \tag{III.2.16}$$

Hence

$$\det g_{ij} = g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1 \sin x^2)^2 \end{vmatrix} = (x^1)^4 (\sin x^2)^2 \neq 0 \quad \text{identically.} \tag{III.2.17}$$

Also

$$\begin{aligned} g^{11} &= \frac{\text{cofactor of } g_{11} \text{ in } \det g_{ij}}{\det g_{ij}} \\ &= \frac{(-1)^{1+1} \begin{vmatrix} (x^1)^2 & 0 \\ 0 & (x^1 \sin x^2)^2 \end{vmatrix}}{(x^1)^4 (\sin x^2)^2} \\ &= 1. \end{aligned}$$

Similarly

$$\begin{aligned} g^{22} &= \frac{\text{cofactor of } g_{22} \text{ in } \det g_{ij}}{\det g_{ij}} \\ &= \frac{(-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 0 & (x^1 \sin x^2)^2 \end{vmatrix}}{(x^1)^4 (\sin x^2)^2} \\ &= \frac{1}{(x^1)^2}, \\ g^{33} &= \frac{\text{cofactor of } g_{33} \text{ in } \det g_{ij}}{\det g_{ij}} \\ &= \frac{(x^1)^2}{(x^1)^2 (x^1 \sin x^2)^2} \\ &= \frac{1}{(x^1 \sin x^2)^2} \end{aligned}$$

and

$$g^{ij} = 0, \quad \text{for } i \neq j, \quad \text{since} \quad g_{ij} = 0 \quad \text{for} \quad i \neq j.$$

Now
$$\frac{\partial g_{11}}{\partial x^1} = 0, \frac{\partial g_{22}}{\partial x^1} = 2x^1, \frac{\partial g_{33}}{\partial x^1} = 2x^1 (\sin x^2)^2, \frac{\partial g_{11}}{\partial x^2} = 0,$$
$$\frac{\partial g_{22}}{\partial x^2} = 0, \frac{\partial g_{33}}{\partial x^2} = 2 (x^1)^2 \sin x^2 \cos x^2,$$
$$\frac{\partial g_{11}}{\partial x^3} = 0, \frac{\partial g_{22}}{\partial x^3} = 0 \quad \text{and} \quad \frac{\partial g_{33}}{\partial x^3} = 0.$$

All other partial derivatives of g_{ij} vanish identically.

From
$$[i\,j, k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

it follows that if all i, j, k are different then $[i\,j, k]$ vanishes identically.

Now
$$[i\,i, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^k}\right)$$
$$= \begin{cases} \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i}, & \text{if } i = k \\ -\frac{1}{2}\frac{\partial g_{ii}}{\partial x^k}, & \text{if } i \neq k \end{cases}$$

and
$$[i\,j, i] = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^i}\right)$$
$$= \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}$$
$$= [j\,i, i]$$

Therefore
$$[1\,1, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^1} = 0,$$
$$[2\,2, 2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^2} = 0,$$
$$[3\,3, 3] = \frac{1}{2}\frac{\partial g_{33}}{\partial x^3} = 0,$$
$$[1\,1, 2] = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0,$$
$$[1\,1, 3] = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^3} = 0,$$
$$[2\,2, 1] = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = -x^1,$$
$$[2\,2, 3] = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^3} = 0,$$
$$[3\,3, 1] = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = -x^1 (\sin x^2)^2,$$

$$
\begin{aligned}
[3\,3,2] &= -\frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = -\left(x^1\right)^2 \sin x^2 \cos x^2,\\
[1\,2,1] &= [2\,1,1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0,\\
[1\,3,1] &= [3\,1,1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^3} = 0,\\
[2\,1,2] &= [1\,2,2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = x^1,\\
[2\,3,2] &= [3\,2,2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^3} = 0,\\
[3\,1,3] &= [1\,3,3] = \frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = x^1\left(\sin x^2\right)^2,\\
[3\,2,3] &= [2\,3,3] = \frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = \left(x^1\right)^2 \sin x^2 \cos x^2.
\end{aligned}
$$

Also

$$
\left\{ \begin{matrix} l \\ 1\,1 \end{matrix} \right\} = g^{lk}[1\,1,k] = 0, \text{ since } [1\,1,k] = 0 \text{ for all } k = 1,2,3;
$$

$$
\begin{aligned}
\left\{ \begin{matrix} l \\ 2\,2 \end{matrix} \right\} &= g^{lk}[2\,2,k]\\
&= g^{l1}[2\,2,1] + g^{l2}[2\,2,2] + g^{l3}[2\,2,3]\\
&= g^{l1}\left(-x^1\right) + 0 + 0\\
&= \begin{cases} -x^1 & \text{if } l = 1\\ 0 & \text{otherwise;} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\left\{ \begin{matrix} l \\ 3\,3 \end{matrix} \right\} &= g^{lk}[3\,3,k]\\
&= g^{l1}[3\,3,1] + g^{l2}[3\,3,2] + g^{l3}[3\,3,3]\\
&= -g^{l1}x^1\left(\sin x^1\right)^2 - g^{l2}\left(x^1\right)^2 \sin x^2 \cos x^2 + 0\\
&= \begin{cases} -x^1\left(\sin x^1\right)^2, & \text{if } l = 1\\ -\sin x^2 \cos x^2, & \text{if } l = 2\\ 0, & \text{if } l = 3 \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\left\{ \begin{matrix} l \\ i\,l \end{matrix} \right\} &= g^{lk}[i\,l,k]\\
&= g^{ll}[i\,l,l], \qquad \text{since } g^{lk} = 0 \text{ for } l \neq k\\
&= \left\{ \begin{matrix} l \\ l\,i \end{matrix} \right\}.
\end{aligned}
$$

Hence

$$
\left\{ \begin{matrix} l \\ l\,1 \end{matrix} \right\} = \left\{ \begin{matrix} l \\ 1\,l \end{matrix} \right\} = g^{ll}[1\,l,l]
$$

$$= \begin{cases} 0, & \text{if } l = 1 \\ \frac{1}{x^1}, & \text{if } l = 2, 3 \end{cases},$$

$$\begin{Bmatrix} l \\ l\ 2 \end{Bmatrix} = \begin{Bmatrix} l \\ 2\ l \end{Bmatrix} = g^{ll}\,[2\,l, l]$$

$$= \begin{cases} 0, & \text{if } l = 1, 2 \\ \cot x^2, & \text{if } l = 3 \end{cases},$$

$$\begin{Bmatrix} l \\ l\ 3 \end{Bmatrix} = \begin{Bmatrix} l \\ 3\ l \end{Bmatrix} = g^{ll}\,[3\,l, l] = 0.$$

All other Christoffel symbols of first and second kind vanish identically.

Example 18. If a^{ij} are components of a symmetric tensor prove that

$$a^{jk}\,[i\,j, k] = \frac{1}{2} a^{jk} \frac{\partial g_{jk}}{\partial x^i}$$

where g_{ik} are components of the metric tensor of the Riemannian space.

We know that

$$\frac{\partial g_{jk}}{\partial x^i} = [j\,i, k] + [k\,i, j]\,.$$

Therefore

$$\begin{aligned} \frac{1}{2} a^{jk} \frac{\partial g_{jk}}{\partial x^i} &= \frac{1}{2} a^{jk} \left([j\,i, k] + [k\,i, j]\right) \\ &= \frac{1}{2} \left(a^{jk}\,[j\,i, k] + a^{jk}\,[k\,i, j]\right) \\ &= \frac{1}{2} \left(a^{jk}\,[j\,i, k] + a^{kj}\,[j\,i, k]\right) \\ &\qquad \text{(interchanging the dummy indices } j \text{ and } k \text{ in the last term)} \\ &= \frac{1}{2} \times 2a^{jk}\,[j\,i, k]\,, \qquad \text{(since } a^{jk} \text{ is symmetric)} \\ &= a^{jk}\,[i\,j, k]\,, \qquad \text{(since } [j\,i, k] = [i\,j, k]). \end{aligned}$$

Example 19. If A^{ijk} is a skew-symmetric tensor, show that

$$A^{ijk} \begin{Bmatrix} l \\ i\ j \end{Bmatrix} = A^{ijk} \begin{Bmatrix} l \\ j\ k \end{Bmatrix} = A^{ijk} \begin{Bmatrix} l \\ i\ k \end{Bmatrix} = 0.$$

We see

$$A^{ijk} \begin{Bmatrix} l \\ i\ j \end{Bmatrix} = A^{jik} \begin{Bmatrix} l \\ j\ i \end{Bmatrix},$$

(interchanging dummy indices i and j)

$$= -A^{ijk}\left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\}$$

(since A^{ijk} is skew symmetric and the Christoffel symbol of second kind is symmetric in second two indices).

This implies

$$A^{ijk}\left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} = 0.$$

Example 20. If

$$\bar{\Gamma}^i_{jk} = \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\} + \delta^i_j A_k + \delta^i_k A_j$$

where A_i is a covariant vector and T^{ij} is any skew-symmetric tensor show that

$$T^{jk}\bar{\Gamma}^i_{jk} = 0.$$

We have

$$\begin{aligned} \bar{\Gamma}^i_{kj} &= \left\{ \begin{matrix} i \\ k\ j \end{matrix} \right\} + \delta^i_k A_j + \delta^i_j A_k \\ &= \bar{\Gamma}^i_{jk}. \end{aligned} \tag{III.2.18}$$

Now

$$\begin{aligned} T^{jk}\bar{\Gamma}^i_{jk} &= T^{kj}\bar{\Gamma}^i_{kj} \text{ (interchanging the dummy indices } j \text{ and } k) \\ &= -T^{jk}\bar{\Gamma}^i_{jk} \end{aligned}$$

(using (III.2.18) and using the skew symmetry property of T^{jk}).

This implies

$$T^{jk}\bar{\Gamma}^i_{jk} = 0.$$

Theorem 2.1 : The maximum number of independent components of the Christoffel symbols in an n-dimensional Riemannian space is $\frac{n^2(n+1)}{2}$.

Proof : Since i and j can run from 1 to n, g_{ij} has n^2 components. Among these n has equal suffixes (i.e. of the type g_{ii}). Since g_{ij} is symmetric the rest $n^2 - n$ terms with unequal indices reduces to $\frac{n^2-n}{2}$ independent components. Combining both the cases g_{ij} have at most $n + \frac{n^2-n}{2} = \frac{n(n+1)}{2}$ components.

Consequently the number of independent components of $\frac{\partial g_{ij}}{\partial x^l}$ is $n \times \frac{n(n+1)}{2} = \frac{n^2(n+1)}{2}$ (since l varies from 1 to n).

$[ij,k]$ and $\left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\}$ are formed by linear combinations of $\frac{\partial g_{ij}}{\partial x^l}$. Hence it follows that the maximum number of independent components of Christoffel symbols is same as that of $\frac{\partial g_{ij}}{\partial x^l}$, i.e., $\frac{n^2(n+1)}{2}$.

Theorem 2.2 : Christoffel symbols vanish identically if and only if g_{ij}'s are constants.

Proof : First we assume that g_{ij}'s are constant. Then for all $l = 1, 2, \ldots, n$

$$\frac{\partial g_{ij}}{\partial x^l} = 0.$$

Therefore for all $p, q, r = 1, 2, \ldots, n$

$$[pq,r] = \frac{1}{2}\left(\frac{\partial g_{qr}}{\partial x^p} + \frac{\partial g_{pr}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^r}\right) = 0.$$

And

$$\left\{ \begin{matrix} l \\ p\ q \end{matrix} \right\} = g^{lr}[pq,r] = g^{lr} \times 0 = 0.$$

Conversely, let all the Christoffel symbols vanish identically. Then for all $i, j, k = 1, 2, \ldots, n$

$$[ij,k] = 0 .$$

Therefore

$$\frac{\partial g_{ij}}{\partial x^k} = ([ik,j] + [jk,i]) = 0.$$

This is true for any $i, j, k = 1, 2, \ldots, n$. This implies g_{ij}'s are independent of x^i's, i.e., g_{ij}'s are constants.

An important note. In an n-dimensional Euclidean space E_n with rectangular Cartesian coordinate system

$$g_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\frac{\partial g_{ij}}{\partial x^k} = 0, \qquad \text{for all} \quad i, j, k = 1, 2, \ldots, n.$$

This is why, in n-dimensional Euclidean space E_n with rectangular Cartesian coordinates, all the Christoffel symbols vanish identically.

Example 21. Evaluate Christoffel symbols for Riemannian spaces where $g_{ij} = 0$ if $i \neq j$.

If $i = j = k$, (i.e., all the three indices are same)

$$[ij,k] = [ii,i] = \frac{1}{2}\left(\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}\right) = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i}. \qquad \text{(III.2.19)}$$

If $i = j$, $i \neq k$ (i.e., the first two indices are same but the third is different)

$$[ij, k] = [ii, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^k}\right) = -\frac{1}{2}\frac{\partial g_{ii}}{\partial x^k}. \tag{III.2.20}$$

If $i = k$, $i \neq j$ (i.e., the first and third are same or the second and third are same, the rest being different)

$$[ij, k] = [ij, i] = [ji, i] = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^i}\right) = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}. \tag{III.2.21}$$

If $i \neq j$, $j \neq k$, $k \neq i$ (i.e., all three indices are different)

$$[ij, k] = 0. \tag{III.2.22}$$

Now

$$\begin{aligned} g &= \det g_{ij} \\ &= \begin{vmatrix} g_{11} & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & 0 \\ 0 & 0 & \cdots & g_{nn} \end{vmatrix} \\ &= g_{11}g_{22}\cdots g_{nn}. \end{aligned}$$

So

$$\begin{aligned} g^{ij} &= \frac{\text{cofactor of } g_{ij} \text{ in } g}{g} \\ &= \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{g_{ii}} & \text{if } i = j. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \left\{ \begin{matrix} l \\ i\,j \end{matrix} \right\} &= g^{lk}[ij, k] \\ &= \begin{cases} 0 & \text{if } l \neq k \\ g^{kk}[ij, k] & \text{if } l = k. \end{cases} \end{aligned}$$

Therefore if $i = j = k$, then

$$\begin{aligned} \left\{ \begin{matrix} k \\ i\,j \end{matrix} \right\} &= \left\{ \begin{matrix} i \\ i\,i \end{matrix} \right\} \\ &= g^{ii}[ii, i] \\ &= \frac{1}{g_{ii}}\frac{1}{2}\frac{\partial g_{ii}}{\partial x^i}, \qquad \text{using (III.2.19)} \\ &= \frac{1}{2}\frac{\partial}{\partial x^i}(\log g_{ii}) \\ &= \frac{\partial}{\partial x^i}\log\sqrt{g_{ii}}\,; \end{aligned}$$

If $i = j$, $i \neq k$

$$\begin{Bmatrix} k \\ i\ j \end{Bmatrix} = \begin{Bmatrix} k \\ i\ i \end{Bmatrix}$$
$$= g^{km}[im, k]$$
$$= \frac{1}{g_{kk}}\left(-\frac{1}{2}\frac{\partial g_{ii}}{\partial x^k}\right), \qquad \text{using (III.2.20)}$$
$$= -\frac{1}{2g_{kk}}\frac{\partial g_{ii}}{\partial x^k}$$

and if $i = k$, $j \neq k$,

$$\begin{Bmatrix} k \\ i\ j \end{Bmatrix} = \begin{Bmatrix} i \\ i\ j \end{Bmatrix}$$
$$= g^{im}[ij, i]$$
$$= \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^j}$$
$$= \frac{\partial}{\partial x^j}(\log\sqrt{g_{ii}}).$$

Lastly, if $i \neq j$, $j \neq k$, $k \neq i$:

$$\begin{Bmatrix} k \\ i\ j \end{Bmatrix} = 0.$$

III.2.4 Law of Transformation of Christoffel Symbols of First Kind

We consider an n-dimensional Riemannian space V_n with fundamental metric tensor g_{ij} in (x^i) coordinate system. Then

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right). \qquad \text{(III.2.23)}$$

We consider a coordinate transformation given by

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, x^2, \ldots, x^n) \qquad (\alpha = 1, 2, \ldots n) \qquad \text{(III.2.24)}$$

and let $\bar{g}_{\alpha\beta}$ denote the components of the fundamental metric tensor in $(\bar{x}^\alpha)$ coordinate system.

Since $\bar{g}_{\alpha\beta}$ are the components of a covariant tensor of type $(0, 2)$ we have

$$\bar{g}_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}. \qquad \text{(III.2.25)}$$

Also in $(\bar{x}^\alpha)$ coordinate system

$$\overline{[\alpha\beta, \gamma]} = \frac{1}{2}\left(\frac{\partial \bar{g}_{\beta\gamma}}{\partial \bar{x}^\alpha} + \frac{\partial \bar{g}_{\alpha\gamma}}{\partial \bar{x}^\beta} - \frac{\partial \bar{g}_{\alpha\beta}}{\partial \bar{x}^\gamma}\right). \qquad \text{(III.2.26)}$$

Differentiating both sides of (III.2.25) with respect to $\bar{x}^\gamma$ we obtain

$$\frac{\partial \bar{g}_{\alpha\beta}}{\partial \bar{x}^\gamma} = \frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} + g_{ij}\frac{\partial^2 x^i}{\partial \bar{x}^\gamma \partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} + g_{ij}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial^2 x^j}{\partial \bar{x}^\gamma \partial \bar{x}^\beta}. \quad \text{(III.2.27)}$$

Changing α, β, γ and i, j, k cyclically from the above equation we get the following two :

$$\frac{\partial \bar{g}_{\beta\gamma}}{\partial \bar{x}^\alpha} = \frac{\partial g_{jk}}{\partial x^i}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{jk}\frac{\partial^2 x^j}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{jk}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial^2 x^k}{\partial \bar{x}^\alpha \partial \bar{x}^\gamma} \quad \text{(III.2.28)}$$

and

$$\frac{\partial \bar{g}_{\gamma\alpha}}{\partial \bar{x}^\beta} = \frac{\partial g_{ki}}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + g_{ki}\frac{\partial^2 x^k}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + g_{ki}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}. \quad \text{(III.2.29)}$$

Using (III.2.27), (III.2.28) and (III.2.29) we get from (III.2.26)

$$\begin{aligned}
\overline{[\alpha\beta,\gamma]} &= \frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial x^i}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{jk}\frac{\partial^2 x^j}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{jk}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial^2 x^k}{\partial \bar{x}^\alpha \partial \bar{x}^\gamma}\right.\\
&\quad + \frac{\partial g_{ki}}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + g_{ki}\frac{\partial^2 x^k}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + g_{ki}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}\\
&\quad \left. - \frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} - g_{ij}\frac{\partial^2 x^i}{\partial \bar{x}^\gamma \partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} - g_{ij}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial^2 x^j}{\partial \bar{x}^\gamma \partial \bar{x}^\beta}\right\}\\
&= \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\\
&\quad + \frac{1}{2}\left[g_{jk}\frac{\partial^2 x^j}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{jk}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial^2 x^k}{\partial \bar{x}^\alpha \partial \bar{x}^\gamma}\right.\\
&\quad + g_{ki}\frac{\partial^2 x^k}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + g_{ki}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}\\
&\quad \left. - g_{jk}\frac{\partial^2 x^k}{\partial \bar{x}^\gamma \partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} - g_{ki}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial^2 x^k}{\partial \bar{x}^\gamma \partial \bar{x}^\beta}\right]
\end{aligned}$$

(replacing the dummy index j by k in the last term and replacing the dummy index i by k in its preceeding term)

$$\begin{aligned}
&= [ij,k]\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + \frac{1}{2}\left(g_{jk}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial^2 x^j}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} + g_{ki}\frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}\frac{\partial x^k}{\partial \bar{x}^\gamma}\right)\\
&= [ij,k]\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + \frac{1}{2}\left(g_{ji}\frac{\partial x^i}{\partial \bar{x}^\gamma}\frac{\partial^2 x^j}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} + g_{ji}\frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\gamma}\right)
\end{aligned}$$

(replacing the dummy index k by i in the second term and k by j in the last term)

$$= [ij,k]\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{ij}\frac{\partial x^i}{\partial \bar{x}^\gamma}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha} \quad \text{(III.2.30)}$$

(taking $g_{ij} = g_{ji}$ in the second term and interchanging the dummy indices i and j in the last term).

This is the required law of transformation of the components of Christoffel symbol of first kind.

Note : The presence of the second term in the right hand side of (III.2.30) suggests that the Christoffel symbol of first kind is not a tensor. If the said term did not exist $[ij,k]$ would be a tensor of type $(0,3)$.

III.2.5 Law of Transformation of Christoffel Symbols of Second Kind

We have already proved

$$\overline{[\alpha\beta,\gamma]} = [ij,k]\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{ij}\frac{\partial x^i}{\partial \bar{x}^\gamma}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}. \tag{III.2.31}$$

Multiplying both sides of the above relation by $\bar{g}^{\gamma\sigma}$ we get

$$\bar{g}^{\gamma\sigma}\overline{[\alpha\beta,\gamma]} = [ij,k]\,\bar{g}^{\gamma\sigma}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{ij}\bar{g}^{\gamma\sigma}\frac{\partial x^i}{\partial \bar{x}^\gamma}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}$$

or,

$$\begin{aligned}
\overline{\left\{\begin{matrix}\sigma\\ \alpha\ \beta\end{matrix}\right\}} &= [ij,k]\,g^{lm}\frac{\partial \bar{x}^\gamma}{\partial x^l}\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + g_{ij}g^{lm}\frac{\partial \bar{x}^\gamma}{\partial x^l}\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial x^i}{\partial \bar{x}^\gamma}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}\\
&= [ij,k]\,g^{lm}\delta^k_l\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} + g_{ij}g^{lm}\delta^i_l\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}\\
&= [ij,k]\,g^{mk}\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} + g_{ij}g^{im}\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}\\
&= \left\{\begin{matrix}m\\ i\ j\end{matrix}\right\}\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} + \delta^m_j\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}\\
&= \left\{\begin{matrix}m\\ i\ j\end{matrix}\right\}\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} + \frac{\partial \bar{x}^\sigma}{\partial x^j}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}.
\end{aligned} \tag{III.2.32}$$

Note : The presence of the second term in the right hand side of the above relation asserts that $\left\{\begin{matrix}l\\ i\ j\end{matrix}\right\}$ is not a tensor. If it had vanished, $\left\{\begin{matrix}l\\ i\ j\end{matrix}\right\}$ would have been components of a tensor of type $(1,2)$.

Note : If we consider a particular type of coordinate transformation satisfying

$$\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha} = 0$$

for all $j, \alpha, \beta = 1, 2, \ldots, n$ the Christoffel symbols of 1st and 2nd kind transform like tensors.

Example 22. Discuss the law of transformation of Christoffel symbols for a coordinate transformation given by

$$\bar{x}^l = a^l_i x^i + b^l$$

where a_i^l and b^l are constants.

Given

$$\bar{x}^l = a_i^l x^i + b^l. \tag{III.2.33}$$

Differentiating partially with respect to x^i

$$\frac{\partial \bar{x}^l}{\partial x^i} = a_i^l. \tag{III.2.34}$$

Differentiating again

$$\frac{\partial^2 \bar{x}^l}{\partial x^j \partial x^i} = 0.$$

Hence for such type of transformations Christoffel symbols behave like tensors.

Note : Multiplying both the sides of (III.2.32) by $\dfrac{\partial x^r}{\partial \bar{x}^\sigma}$ we get

$$\frac{\partial x^r}{\partial \bar{x}^\tau}\overline{\begin{Bmatrix} \sigma \\ \alpha\ \beta \end{Bmatrix}} = \begin{Bmatrix} m \\ i\ j \end{Bmatrix} \frac{\partial x^r}{\partial \bar{x}^\sigma}\frac{\partial \bar{x}^\sigma}{\partial x^m}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta} + \frac{\partial x^r}{\partial \bar{x}^\sigma}\frac{\partial \bar{x}^\sigma}{\partial x^j}\frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha}$$

or,

$$\delta_j^r \frac{\partial^2 x^j}{\partial \bar{x}^\beta \partial \bar{x}^\alpha} = \frac{\partial x^r}{\partial \bar{x}^\sigma}\overline{\begin{Bmatrix} \sigma \\ \alpha\ \beta \end{Bmatrix}} - \begin{Bmatrix} m \\ i\ j \end{Bmatrix}\delta_m^r \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}$$

or,

$$\frac{\partial^2 x^r}{\partial \bar{x}^\beta \partial \bar{x}^\alpha} = \frac{\partial x^r}{\partial \bar{x}^\sigma}\overline{\begin{Bmatrix} \sigma \\ \alpha\ \beta \end{Bmatrix}} - \begin{Bmatrix} r \\ i\ j \end{Bmatrix}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}. \tag{III.2.35}$$

This shows that the second order partial derivatives of x^r with respect to $\bar{x}^\alpha$ can be expressed in terms of derivatives of first order and Christoffel symbol of second kind.

In a similar manner $\dfrac{\partial^2 \bar{x}^\alpha}{\partial x^j \partial x^k}$ can be expressed in terms of derivatives of first order and Christoffel symbol of second kind as

$$\frac{\partial^2 \bar{x}^\alpha}{\partial x^k \partial x^i} = \frac{\partial \bar{x}^\alpha}{\partial x^r}\begin{Bmatrix} r \\ i\ k \end{Bmatrix} - \overline{\begin{Bmatrix} \alpha \\ \sigma\ r \end{Bmatrix}}\frac{\partial \bar{x}^r}{\partial x^i}\frac{\partial \bar{x}^\sigma}{\partial x^k}. \tag{III.2.36}$$

Note : Considering three different coordinate systems it can be easily shown that applying the law of transformation of Christoffel symbols from first system to third system is identical as from first system to second system. This means the said properties are transitive, which is expressed by saying 'Christoffel symbols form a group'.

Exercises

Exercise 6 . Calculate the Christoffel symbols $\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}, \begin{Bmatrix} 2 \\ 1\ 2 \end{Bmatrix}$ in a 3–dimensional Riemannian space in which the distance ds is given by

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (dx^3)^2.$$

Exercise 7 . Find the non-vanishing Christoffel symbols of the second kind for V_2 with line element

$$ds^2 = \left(dx^1\right)^2 + \sin^2 x^1 \left(dx^2\right)^2 .$$

Exercise 8 . Calculate the Christoffel symbols in cylindrical coordinates (u, ϕ, z).
[Hints : Put $u = x^1$, $\phi = x^2$, $z = x^3$.
Then $ds^2 = g_{ij}dx^i dx^j = \left(dx^1\right)^2 + \left(x^1\right)^2 \left(dx^2\right)^2 + \left(dx^3\right)^2$.]

Exercise 9 . Find the Christoffel symbol $\left\{ {1 \atop 2\,2} \right\}$ in the Euclidean space with spherical polar coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

III.3 Covariant Differentiation of Tensors

In section (II.3) above we have shown that the partial derivatives of a tensor of type $(0,0)$ is a tensor of type $(0,1)$ whereas the partial derivatives of a tensor of type $(0,1)$ is not a tensor.

In this section we are going to introduce a new type of differentiation called covariant differentiation which is operated on tensors and again produce another tensor.

We start with covariant differentiation of vectors.

III.3.1 Covariant Differentiation of a Covariant Vector

Let us consider a Riemannian space of dimension n with a coordinate system (x^i) defined on it. Let g_{ij} be the components of fundamental metric tensor. Let $\bar{g}_{\alpha\beta}$ be its components in another coordinate system $(\bar{x}^\alpha)$ defined on the same space.

Let λ_i be the components of a covariant vector. Then the law of transformation is given by

$$\bar{\lambda}_\alpha = \lambda_i \frac{\partial x^i}{\partial \bar{x}^\alpha}. \tag{III.3.1}$$

Differentiating partially with respect to $\bar{x}^\beta$ we get

$$\begin{aligned} \frac{\partial \bar{\lambda}_\alpha}{\partial \bar{x}^\beta} &= \frac{\partial \lambda_i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial x^i}{\partial \bar{x}^\alpha} + \lambda_i \frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\alpha} \\ &= \frac{\partial \lambda_i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial x^i}{\partial \bar{x}^\alpha} + \lambda_i \left(\frac{\partial x^i}{\partial \bar{x}^\sigma}\overline{\left\{ {\sigma \atop \alpha\,\beta} \right\}} - \left\{ {i \atop j\,k} \right\}\frac{\partial x^j}{\partial \bar{x}^\alpha}\frac{\partial x^k}{\partial \bar{x}^\beta} \right), \end{aligned}$$

using (III.2.35)

or,

$$\frac{\partial \bar{\lambda}_\alpha}{\partial \bar{x}^\beta} - \lambda_i \frac{\partial x^i}{\partial \bar{x}^\sigma}\overline{\left\{ {\sigma \atop \alpha\,\beta} \right\}} = \frac{\partial \lambda_i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial x^i}{\partial \bar{x}^\alpha} - \lambda_l \left\{ {l \atop i\,k} \right\}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^k}{\partial \bar{x}^\beta}$$

(replacing the dummy index i by l and j by i in the last term of the right hand side)

$$= \left(\frac{\partial \lambda_i}{\partial x^k} - \lambda_l \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\}\right) \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial x^k}{\partial \bar{x}^\beta}$$

or,

$$\frac{\partial \bar{\lambda}_\alpha}{\partial \bar{x}^\beta} - \bar{\lambda}_\sigma \overline{\left\{ \begin{matrix} \sigma \\ \alpha\ \beta \end{matrix} \right\}} = \left(\frac{\partial \lambda_i}{\partial x^k} - \lambda_l \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\}\right) \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial x^k}{\partial \bar{x}^\beta}. \tag{III.3.2}$$

If we write

$$\bar{\lambda}_{\alpha,\beta} = \frac{\partial \bar{\lambda}_\alpha}{\partial \bar{x}^\beta} - \bar{\lambda}_\sigma \overline{\left\{ \begin{matrix} \sigma \\ \alpha\ \beta \end{matrix} \right\}} \tag{III.3.3}$$

and

$$\lambda_{i,k} = \frac{\partial \lambda_i}{\partial x^k} - \lambda_l \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} \tag{III.3.4}$$

then (III.3.2) takes the form

$$\bar{\lambda}_{\alpha,\beta} = \lambda_{i,k} \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial x^k}{\partial \bar{x}^\beta}. \tag{III.3.5}$$

The above equation confirms that $\lambda_{i,k}$ are the components of a tensor of type $(0,2)$.

The set of n^2 functions $\frac{\partial \lambda_i}{\partial x^k} - \lambda_l \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\}$ is said to define the covariant x^k derivative (with respect to the fundamental tensor g_{ij}) of the covariant tensor λ_i and is denoted by $\lambda_{i,k}$.

Note : Thus we observe that the covariant derivative of a tensor of type $(0,1)$ is a tensor of type $(0,2)$.

III.3.2 Covariant Differentiation of a Contravariant Vector

Let us consider a contravariant vector whose components in (x^i) and $(\bar{x}^\alpha)$ coordinate systems are given by λ^i and $\bar{\lambda}^\alpha$ respectively.
Then the law of transformation is given by

$$\bar{\lambda}^\alpha = \lambda^i \frac{\partial \bar{x}^\alpha}{\partial x^i}. \tag{III.3.6}$$

Differentiating partially with respect to $\bar{x}^\beta$ we get

$$\frac{\partial \bar{\lambda}^\alpha}{\partial \bar{x}^\beta} = \frac{\partial \lambda^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\alpha}{\partial x^i} + \lambda^i \frac{\partial^2 \bar{x}^\alpha}{\partial x^k \partial x^i} \frac{\partial x^k}{\partial \bar{x}^\beta}$$

$$= \frac{\partial \lambda^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\alpha}{\partial x^i} + \lambda^i \frac{\partial x^k}{\partial \bar{x}^\beta} \left(\frac{\partial \bar{x}^\alpha}{\partial x^r} \left\{ \begin{matrix} r \\ i\ k \end{matrix} \right\} - \overline{\left\{ \begin{matrix} \alpha \\ \sigma\ \gamma \end{matrix} \right\}} \frac{\partial \bar{x}^\gamma}{\partial x^i} \frac{\partial \bar{x}^\sigma}{\partial x^k} \right),$$

using (III.2.36)

$$= \frac{\partial \lambda^r}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\alpha}{\partial x^r} + \lambda^i \frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\alpha}{\partial x^r}\begin{Bmatrix} r \\ i\ k \end{Bmatrix} - \lambda^i \begin{Bmatrix} \alpha \\ \sigma\ \gamma \end{Bmatrix}\frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\gamma}{\partial x^i}\frac{\partial \bar{x}^\sigma}{\partial x^k}$$

(replacing the dummy index i by r in the first term)

$$= \left(\frac{\partial \lambda^r}{\partial x^k} + \lambda^i \begin{Bmatrix} r \\ i\ k \end{Bmatrix}\right)\frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\alpha}{\partial x^r} - \lambda^i \begin{Bmatrix} \alpha \\ \sigma\ \gamma \end{Bmatrix}\frac{\partial \bar{x}^\gamma}{\partial x^i}\delta^\sigma_\beta$$

$$= \left(\frac{\partial \lambda^r}{\partial x^k} + \lambda^i \begin{Bmatrix} r \\ i\ k \end{Bmatrix}\right)\frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\alpha}{\partial x^r} - \lambda^i \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix}\frac{\partial \bar{x}^\gamma}{\partial x^i}.$$

Therefore

$$\frac{\partial \bar{\lambda}^\alpha}{\partial \bar{x}^\beta} + \lambda^i \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix}\frac{\partial \bar{x}^\gamma}{\partial x^i} = \left(\frac{\partial \lambda^r}{\partial x^k} + \lambda^i \begin{Bmatrix} r \\ i\ k \end{Bmatrix}\right)\frac{\partial x^k}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\alpha}{\partial x^r}. \tag{III.3.7}$$

Writing

$$\bar{\lambda}^\alpha_{,\beta} = \frac{\partial \bar{\lambda}^\alpha}{\partial \bar{x}^\beta} + \bar{\lambda}^\gamma \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} \tag{III.3.8}$$

and

$$\lambda^r_{,k} = \frac{\partial \lambda^r}{\partial x^k} + \lambda^i \begin{Bmatrix} r \\ i\ k \end{Bmatrix} \tag{III.3.9}$$

the equation (III.3.7) takes the form

$$\bar{\lambda}^\alpha_{,\beta} = \lambda^r_{,k}\frac{\partial \bar{x}^\alpha}{\partial x^r}\frac{\partial x^k}{\partial \bar{x}^\beta}. \tag{III.3.10}$$

This shows that $\lambda^r_{,k}$ are the components of a tensor of type $(1,1)$. This tensor $\lambda^r_{,k}$ is defined to be the covariant derivative of the contravariant vector with components λ^r.

Note : From the above discussion it follows that the covariant derivative of a tensor of type $(1,0)$ is a tensor of type $(1,1)$.

Note : In (III.3.9) putting $k = r$ we get

$$\lambda^r_{,r} = \frac{\partial \lambda^r}{\partial x^r} + \lambda^i \begin{Bmatrix} r \\ i\ r \end{Bmatrix}$$

$$= \frac{\partial \lambda^r}{\partial x^r} + \lambda^i \frac{\partial}{\partial x^i}(\log \sqrt{g}), \qquad \text{using (III.2.13)}$$

$$= \frac{1}{\sqrt{g}}\left[\sqrt{g}\frac{\partial \lambda^r}{\partial x^r} + \sqrt{g}\lambda^i \frac{\partial}{\partial x^i}\left(\frac{1}{2}\log g\right)\right]$$

$$= \frac{1}{\sqrt{g}}\left[\sqrt{g}\frac{\partial \lambda^i}{\partial x^i} + \lambda^i \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x^i}\right]$$

(replacing the dummy index r by i)

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\lambda^i).$$

Since $\lambda^r_{,k}$ is a tensor of type $(1,1)$, it follows that $\lambda^r_{,r}$ is a tensor of type $(1-1, 1-1) = (0,0)$, *i.e.*, $\lambda^r_{,r}$ is a scalar.

III.3.3 Covariant Differentiation of Tensors of Type (0,2)

Let us consider a tensor of type $(0,2)$ whose components in (x^i) and $(\bar{x}^\alpha)$ coordinate systems are denoted by λ_{ij} and $\bar{\lambda}_{\alpha\beta}$ respectively, where $i,j,\alpha,\beta = 1,2,\ldots,n-1,n$.

We have,

$$\bar{\lambda}_{\alpha\beta} = \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\lambda_{ij}. \tag{III.3.11}$$

Differentiating both sides of (III.3.11) with respect to $\bar{x}^\gamma$ we get

$$\begin{aligned}
\frac{\partial \bar{\lambda}_{\alpha\beta}}{\partial \bar{x}^\gamma} &= \frac{\partial^2 x^i}{\partial \bar{x}^\gamma \partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\lambda_{ij} + \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial^2 x^j}{\partial \bar{x}^\gamma \partial \bar{x}^\beta}\lambda_{ij} + \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial \lambda_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\gamma} \\
&= \lambda_{ij}\frac{\partial x^j}{\partial \bar{x}^\beta}\left[\frac{\partial x^i}{\partial \bar{x}^\sigma}\overline{\begin{Bmatrix} \sigma \\ \gamma\ \alpha \end{Bmatrix}} - \begin{Bmatrix} i \\ l\ m \end{Bmatrix}\frac{\partial x^l}{\partial \bar{x}^\alpha}\frac{\partial x^m}{\partial \bar{x}^\gamma}\right] \\
&\quad + \lambda_{ij}\frac{\partial x^i}{\partial \bar{x}^\alpha}\left[\frac{\partial x^j}{\partial \bar{x}^\sigma}\overline{\begin{Bmatrix} \sigma \\ \gamma\ \beta \end{Bmatrix}} - \begin{Bmatrix} j \\ l\ m \end{Bmatrix}\frac{\partial x^l}{\partial \bar{x}^\gamma}\frac{\partial x^m}{\partial \bar{x}^\beta}\right] \\
&\quad + \frac{\partial \lambda_{ij}}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} \qquad \text{using (III.2.35)} \\
&= \lambda_{ij}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^i}{\partial \bar{x}^\sigma}\overline{\begin{Bmatrix} \sigma \\ \gamma\ \alpha \end{Bmatrix}} - \lambda_{lj}\begin{Bmatrix} l \\ i\ k \end{Bmatrix}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} \\
&\quad + \lambda_{ij}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\sigma}\overline{\begin{Bmatrix} \sigma \\ \gamma\ \beta \end{Bmatrix}} - \lambda_{im}\begin{Bmatrix} m \\ k\ j \end{Bmatrix}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} \\
&\quad + \frac{\partial \lambda_{ij}}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}
\end{aligned}$$

(in the second term we have replaced the dummy index m by k and interchanged the dummy indices i and l; in the fourth term we have replaced the dummy index l by k and interchanged the dummy indices j and m)

$$\begin{aligned}
&= \bar{\lambda}_{\sigma\beta}\overline{\begin{Bmatrix} \sigma \\ \gamma\ \alpha \end{Bmatrix}} - \lambda_{lj}\begin{Bmatrix} l \\ i\ k \end{Bmatrix}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} \\
&\quad + \bar{\lambda}_{\alpha\sigma}\overline{\begin{Bmatrix} \sigma \\ \gamma\ \beta \end{Bmatrix}} - \lambda_{il}\begin{Bmatrix} l \\ k\ j \end{Bmatrix}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} \\
&\quad + \frac{\partial \lambda_{ij}}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}
\end{aligned}$$

(replacing the dummy index m by l in the fourth term).

Therefore

$$\begin{aligned}
&\frac{\partial \bar{\lambda}_{\alpha\beta}}{\partial \bar{x}^\gamma} - \bar{\lambda}_{\sigma\beta}\overline{\begin{Bmatrix} \sigma \\ \gamma\ \alpha \end{Bmatrix}} - \bar{\lambda}_{\alpha\sigma}\overline{\begin{Bmatrix} \sigma \\ \gamma\ \beta \end{Bmatrix}} \\
&= \left(\frac{\partial \lambda_{ij}}{\partial x^k} - \lambda_{lj}\begin{Bmatrix} l \\ i\ k \end{Bmatrix} - \lambda_{il}\begin{Bmatrix} l \\ k\ j \end{Bmatrix}\right)\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}.
\end{aligned} \tag{III.3.12}$$

Putting

$$\bar{\lambda}_{\alpha\beta,\gamma} = \frac{\partial \bar{\lambda}_{\alpha\beta}}{\partial \bar{x}^{\gamma}} - \bar{\lambda}_{\sigma\beta}\overline{\left\{ \begin{matrix} \sigma \\ \gamma\ \alpha \end{matrix} \right\}} - \bar{\lambda}_{\alpha\sigma}\overline{\left\{ \begin{matrix} \sigma \\ \gamma\ \beta \end{matrix} \right\}} \qquad \text{(III.3.13)}$$

and

$$\lambda_{ij,k} = \frac{\partial \lambda_{ij}}{\partial x^k} - \lambda_{lj}\left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} - \lambda_{il}\left\{ \begin{matrix} l \\ k\ j \end{matrix} \right\}. \qquad \text{(III.3.14)}$$

the equation (III.3.12) can be rewritten as

$$\bar{\lambda}_{\alpha\beta,\gamma} = \lambda_{ij,k}\frac{\partial x^i}{\partial \bar{x}^{\alpha}}\frac{\partial x^j}{\partial \bar{x}^{\beta}}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}. \qquad \text{(III.3.15)}$$

This shows that $\lambda_{ij,k}$ are the components of a tensor of type $(0,3)$. This tensor $\lambda_{ij,k}$ is called the covariant derivative of the covariant tensor λ_{ij}.

Note : From the above discussion it follows that the covariant derivative of a tensor of type $(0,2)$ is a tensor of type $(0,3)$.

III.3.4 Covariant Derivative of a Tensor of Type (2,0)

Let us consider a tensor of type $(2,0)$ whose components in (x^i) and $(\bar{x}^{\alpha})$ coordinate systems are λ^{ij} and $\bar{\lambda}^{\alpha\beta}$ respectively. Then the law of transformation is

$$\bar{\lambda}^{\alpha\beta} = \lambda^{ij}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial \bar{x}^{\beta}}{\partial x^j}. \qquad \text{(III.3.16)}$$

Differentiating partially with respect to $\bar{x}^{\gamma}$ we get

$$\begin{aligned}
\frac{\partial \bar{\lambda}^{\alpha\beta}}{\partial \bar{x}^{\gamma}} &= \frac{\partial \lambda^{ij}}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial \bar{x}^{\beta}}{\partial x^j} + \lambda^{ij}\frac{\partial^2 \bar{x}^{\alpha}}{\partial x^k \partial x^i}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\frac{\partial \bar{x}^{\beta}}{\partial x^j} + \lambda^{ij}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial^2 \bar{x}^{\beta}}{\partial x^k \partial x^j}\frac{\partial x^k}{\partial \bar{x}^{\gamma}} \\
&= \frac{\partial \lambda^{ij}}{\partial x^k}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial \bar{x}^{\beta}}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^{\gamma}} + \lambda_{ij}\frac{\partial \bar{x}^{\beta}}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\left(\left\{ \begin{matrix} r \\ k\ i \end{matrix} \right\}\frac{\partial \bar{x}^{\alpha}}{\partial x^r} - \overline{\left\{ \begin{matrix} \alpha \\ \gamma\ \sigma \end{matrix} \right\}}\frac{\partial \bar{x}^{\gamma}}{\partial x^i}\frac{\partial \bar{x}^{\sigma}}{\partial x^k}\right) \\
&\quad + \lambda^{ij}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\left(\left\{ \begin{matrix} r \\ k\ j \end{matrix} \right\}\frac{\partial \bar{x}^{\beta}}{\partial x^r} - \overline{\left\{ \begin{matrix} \beta \\ \gamma\ \sigma \end{matrix} \right\}}\frac{\partial \bar{x}^{\gamma}}{\partial x^j}\frac{\partial \bar{x}^{\sigma}}{\partial x^k}\right) \\
&= \frac{\partial \lambda^{ij}}{\partial x^k}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial \bar{x}^{\beta}}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^{\gamma}} + \lambda^{ij}\frac{\partial \bar{x}^{\alpha}}{\partial x^r}\frac{\partial \bar{x}^{\beta}}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\left\{ \begin{matrix} r \\ k\ i \end{matrix} \right\} \\
&\quad - \lambda^{ij}\frac{\partial \bar{x}^{\beta}}{\partial x^j}\frac{\partial \bar{x}^{\gamma}}{\partial x^i}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\frac{\partial \bar{x}^{\sigma}}{\partial x^k}\overline{\left\{ \begin{matrix} \alpha \\ \gamma\ \sigma \end{matrix} \right\}} \\
&\quad + \lambda^{ij}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial \bar{x}^{\beta}}{\partial x^r}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\left\{ \begin{matrix} r \\ k\ j \end{matrix} \right\} - \lambda^{ij}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial \bar{x}^{\gamma}}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\frac{\partial \bar{x}^{\sigma}}{\partial x^k}\overline{\left\{ \begin{matrix} \beta \\ \gamma\ \sigma \end{matrix} \right\}} \\
&= \frac{\partial \lambda^{ij}}{\partial x^k}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial \bar{x}^{\beta}}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^{\gamma}} + \lambda^{rj}\frac{\partial \bar{x}^{\alpha}}{\partial x^i}\frac{\partial \bar{x}^{\beta}}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^{\gamma}}\left\{ \begin{matrix} i \\ k\ r \end{matrix} \right\} \\
&\quad - \lambda^{ij}\frac{\partial \bar{x}^{\beta}}{\partial x^j}\frac{\partial \bar{x}^{\gamma}}{\partial x^i}\delta^{\sigma}_{\gamma}\overline{\left\{ \begin{matrix} \alpha \\ \gamma\ \sigma \end{matrix} \right\}}
\end{aligned}$$

$$+\lambda^{ir}\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial\bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{k}}{\partial\bar{x}^{\gamma}}\left\{\begin{matrix} j \\ k\ r \end{matrix}\right\} - \lambda^{ij}\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial\bar{x}^{\gamma}}{\partial x^{j}}\delta^{\sigma}_{\gamma}\overline{\left\{\begin{matrix} \beta \\ \gamma\ \sigma \end{matrix}\right\}}$$

(interchanging the dummy indices r and i in the second term and the dummy indices r and j in the fourth term)

$$= \left(\frac{\partial\lambda^{ij}}{\partial x^{k}} + \lambda^{rj}\left\{\begin{matrix} i \\ k\ r \end{matrix}\right\} + \lambda^{ir}\left\{\begin{matrix} j \\ k\ r \end{matrix}\right\}\right)\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial\bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{k}}{\partial\bar{x}^{\gamma}}$$

$$-\lambda^{ij}\frac{\partial\bar{x}^{\beta}}{\partial x^{j}}\frac{\partial\bar{x}^{\sigma}}{\partial x^{i}}\overline{\left\{\begin{matrix} \alpha \\ \gamma\ \sigma \end{matrix}\right\}} - \lambda^{ij}\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial\bar{x}^{\sigma}}{\partial x^{j}}\overline{\left\{\begin{matrix} \beta \\ \gamma\ \sigma \end{matrix}\right\}}$$

or,

$$\frac{\partial\bar{\lambda}^{\alpha\beta}}{\partial\bar{x}^{\gamma}} + \bar{\lambda}^{\sigma\beta}\overline{\left\{\begin{matrix} \alpha \\ \sigma\ \gamma \end{matrix}\right\}} + \bar{\lambda}^{\alpha\sigma}\overline{\left\{\begin{matrix} \beta \\ \gamma\ \sigma \end{matrix}\right\}}$$

$$= \left(\frac{\partial\lambda^{ij}}{\partial x^{k}} + \lambda^{rj}\left\{\begin{matrix} i \\ k\ r \end{matrix}\right\} + \lambda^{ir}\left\{\begin{matrix} j \\ k\ r \end{matrix}\right\}\right)\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial\bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{k}}{\partial\bar{x}^{\gamma}}. \qquad \text{(III.3.17)}$$

Writing

$$\bar{\lambda}^{\alpha\beta}_{\ ,\gamma} = \frac{\partial\bar{\lambda}^{\alpha\beta}}{\partial\bar{x}^{\gamma}} + \bar{\lambda}^{\sigma\beta}\overline{\left\{\begin{matrix} \alpha \\ \sigma\ \gamma \end{matrix}\right\}} + \bar{\lambda}^{\alpha\sigma}\overline{\left\{\begin{matrix} \beta \\ \sigma\ \gamma \end{matrix}\right\}} \qquad \text{(III.3.18)}$$

and

$$\lambda^{ij}_{\ ,k} = \frac{\partial\lambda^{ij}}{\partial x^{k}} + \lambda^{rj}\left\{\begin{matrix} i \\ k\ r \end{matrix}\right\} + \lambda^{ir}\left\{\begin{matrix} j \\ k\ r \end{matrix}\right\}, \qquad \text{(III.3.19)}$$

the equation (III.3.17) takes the form

$$\bar{\lambda}^{\alpha\beta}_{\ ,\gamma} = \lambda^{ij}_{\ ,k}\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial\bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{k}}{\partial\bar{x}^{\gamma}}. \qquad \text{(III.3.20)}$$

The above equation shows that $\lambda^{ij}_{\ ,k}$ are the components of a tensor of type $(2,1)$. This tensor $\lambda^{ij}_{\ ,k}$ is called the covariant derivative of the contravariant tensor with components λ^{ij}.

Note : From the above discussion we conclude that the covariant derivative of a tensor of type $(2,0)$ is a tensor of type $(2,1)$.

III.3.5 Covariant Derivative of a Mixed Tensor of Type (1,1)

Let us consider a mixed tensor of type $(1,1)$ whose components in (x^{i}) and $(\bar{x}^{\alpha})$ coordinate systems are A^{i}_{j} and $\bar{A}^{\alpha}_{\beta}$ respectively. The law of transformation is given by

$$\bar{A}^{\alpha}_{\beta} = A^{i}_{j}\frac{\partial\bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial x^{j}}{\partial\bar{x}^{\beta}}. \qquad \text{(III.3.21)}$$

The above equation can be rewritten as

$$\bar{A}^{\alpha}_{\beta}\frac{\partial x^{i}}{\partial\bar{x}^{\alpha}} = A^{i}_{j}\frac{\partial x^{j}}{\partial\bar{x}^{\beta}}. \qquad \text{(III.3.22)}$$

Differentiating both sides of the above equation with respect to $\bar{x}^\gamma$ we get

$$\frac{\partial \bar{A}^\alpha_\beta}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + \bar{A}^\alpha_\beta \frac{\partial^2 x^i}{\partial \bar{x}^\gamma \partial \bar{x}^\alpha} = \frac{\partial A^i_j}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\gamma}\frac{\partial x^j}{\partial \bar{x}^\beta} + A^i_j \frac{\partial^2 x^j}{\partial \bar{x}^\gamma \partial \bar{x}^\beta}.$$

This implies

$$\frac{\partial \bar{A}^\alpha_\beta}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + \bar{A}^\alpha_\beta \left(\frac{\partial x^i}{\partial \bar{x}^\sigma}\overline{\left\{ \begin{matrix} \sigma \\ \gamma\ \alpha \end{matrix} \right\}} - \left\{ \begin{matrix} i \\ l\ j \end{matrix} \right\}\frac{\partial x^l}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\gamma}\right)$$
$$= \frac{\partial A^i_j}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + A^i_j\left(\frac{\partial x^j}{\partial \bar{x}^\sigma}\overline{\left\{ \begin{matrix} \sigma \\ \beta\ \gamma \end{matrix} \right\}} - \left\{ \begin{matrix} j \\ l\ p \end{matrix} \right\}\frac{\partial x^l}{\partial \bar{x}^\beta}\frac{\partial x^p}{\partial \bar{x}^\gamma}\right)$$

or,

$$\frac{\partial \bar{A}^\alpha_\beta}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + \bar{A}^\alpha_\beta \frac{\partial x^i}{\partial \bar{x}^\sigma}\overline{\left\{ \begin{matrix} \sigma \\ \gamma\ \alpha \end{matrix} \right\}} - \bar{A}^\alpha_\beta \left\{ \begin{matrix} i \\ l\ j \end{matrix} \right\}\frac{\partial x^l}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\gamma}$$
$$= \frac{\partial A^i_j}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + A^i_j\frac{\partial x^j}{\partial \bar{x}^\sigma}\overline{\left\{ \begin{matrix} \sigma \\ \beta\ \gamma \end{matrix} \right\}} - A^i_j\left\{ \begin{matrix} j \\ l\ p \end{matrix} \right\}\frac{\partial x^l}{\partial \bar{x}^\beta}\frac{\partial x^p}{\partial \bar{x}^\gamma}$$

or,

$$\frac{\partial \bar{A}^\alpha_\beta}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + \bar{A}^\alpha_\beta \frac{\partial x^i}{\partial \bar{x}^\alpha}\overline{\left\{ \begin{matrix} \alpha \\ \gamma\ \sigma \end{matrix} \right\}} - \left\{ \begin{matrix} i \\ l\ j \end{matrix} \right\}A^p_q\frac{\partial \bar{x}^\alpha}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^\beta}\frac{\partial x^l}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\gamma}$$
$$= \frac{\partial A^i_j}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + \bar{A}^\alpha_\eta\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\eta}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^\sigma}\overline{\left\{ \begin{matrix} \sigma \\ \beta\ \gamma \end{matrix} \right\}} - A^i_j\left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}$$

(by interchanging the dummy indices α and σ in the second term of left hand side, by interchanging the dummy indices j and l and replacing the dummy index p by k in the last term of the right hand side)
or,

$$\frac{\partial \bar{A}^\alpha_\beta}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + \bar{A}^\alpha_\beta\overline{\left\{ \begin{matrix} \alpha \\ \gamma\ \sigma \end{matrix} \right\}}\frac{\partial x^i}{\partial \bar{x}^\alpha} - \bar{A}^\alpha_\eta\frac{\partial x^i}{\partial \bar{x}^\alpha}\delta^\eta_\sigma\overline{\left\{ \begin{matrix} \sigma \\ \beta\ \gamma \end{matrix} \right\}}$$
$$= \frac{\partial A^i_j}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} - A^i_l\left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + \left\{ \begin{matrix} i \\ l\ j \end{matrix} \right\}A^p_q\delta^l_p\frac{\partial x^q}{\partial \bar{x}^\beta}\frac{\partial x^j}{\partial \bar{x}^\gamma}$$

or,

$$\frac{\partial \bar{A}^\alpha_\beta}{\partial \bar{x}^\gamma}\frac{\partial x^i}{\partial \bar{x}^\alpha} + \bar{A}^\alpha_\beta\overline{\left\{ \begin{matrix} \alpha \\ \gamma\ \sigma \end{matrix} \right\}}\frac{\partial x^i}{\partial \bar{x}^\alpha} - \bar{A}^\alpha_\sigma\overline{\left\{ \begin{matrix} \sigma \\ \beta\ \gamma \end{matrix} \right\}}\frac{\partial x^i}{\partial \bar{x}^\alpha}$$
$$= \frac{\partial A^i_j}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} - A^i_l\left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\}\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma} + \left\{ \begin{matrix} i \\ l\ k \end{matrix} \right\}A^l_j\frac{\partial x^j}{\partial \bar{x}^\beta}\frac{\partial x^k}{\partial \bar{x}^\gamma}$$

(replacing the dummy index j by k and q by j in the last term of right hand side). (III.3.23)

Writing

$$\bar{A}^{\alpha}_{\beta,\gamma} = \frac{\partial \bar{A}^{\alpha}_{\beta}}{\partial \bar{x}^{\gamma}} + \bar{A}^{\sigma}_{\beta}\overline{\left\{ \begin{matrix} \alpha \\ \gamma\ \sigma \end{matrix} \right\}} - \bar{A}^{\alpha}_{\sigma}\overline{\left\{ \begin{matrix} \sigma \\ \beta\ \gamma \end{matrix} \right\}} \tag{III.3.24}$$

and

$$A^{i}_{j,k} = \frac{\partial A^{i}_{j}}{\partial x^{k}} + A^{l}_{j}\left\{ \begin{matrix} i \\ l\ k \end{matrix} \right\} - A^{i}_{l}\left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\}, \tag{III.3.25}$$

the equation (III.3.23) takes the form

$$\bar{A}^{\alpha}_{\beta,\gamma} = A^{i}_{j,k}\frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial x^{j}}{\partial \bar{x}^{\beta}}\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}}. \tag{III.3.26}$$

The tensor $A^{i}_{j,k}$ is a mixed tensor of type $(1,2)$ and is called the covariant derivative of the mixed tensor A^{i}_{j} of type $(1,1)$.

Note : The covariant derivative of a tensor of type $(1,1)$ is a tensor of type $(1,2)$.

Example 23. Prove that

$$A^{ij}_{\ ,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{j}}\left(A^{ij}\sqrt{g}\right) + A^{jp}\left\{ \begin{matrix} i \\ j\ p \end{matrix} \right\}$$

where A^{ij} is a tensor of type $(2,0)$.

We have

$$A^{ij}_{\ ,k} = \frac{\partial A^{ij}}{\partial x^{k}} + \left\{ \begin{matrix} i \\ k\ r \end{matrix} \right\}A^{rj} + \left\{ \begin{matrix} j \\ r\ k \end{matrix} \right\}A^{ir}.$$

Putting $k = j$ in the above equation we get

$$\begin{aligned}
A^{ij}_{\ ,j} &= \frac{\partial A^{ij}}{\partial x^{j}} + \left\{ \begin{matrix} i \\ j\ r \end{matrix} \right\}A^{rj} + \left\{ \begin{matrix} j \\ r\ j \end{matrix} \right\}A^{ir} \\
&= \frac{\partial A^{ij}}{\partial x^{j}} + \left\{ \begin{matrix} i \\ j\ r \end{matrix} \right\}A^{rj} + A^{ir}\frac{\partial}{\partial x^{r}}(\log\sqrt{g}) \\
&= \frac{\partial A^{ij}}{\partial x^{j}} + A^{rj}\left\{ \begin{matrix} i \\ j\ r \end{matrix} \right\} + A^{ij}\frac{\partial}{\partial x^{j}}(\log\sqrt{g}) \\
&\qquad \text{(replacing the dummy index } r \text{ by } j \text{ in the last term)} \\
&= \frac{\partial A^{ij}}{\partial x^{j}} + A^{ij}\frac{\partial}{\partial x^{j}}\left(\frac{1}{2}\log g\right) + A^{rj}\left\{ \begin{matrix} i \\ j\ r \end{matrix} \right\} \\
&= \frac{1}{\sqrt{g}}\left(\sqrt{g}\frac{\partial A^{ij}}{\partial x^{j}} + A^{ij}\frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x^{j}}\right) + A^{jp}\left\{ \begin{matrix} i \\ p\ j \end{matrix} \right\} \\
&\qquad \text{(replacing the dummy index } j \text{ by } p \text{ and } r \text{ by } j \text{ in the last term)} \\
&= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{j}}\left(\sqrt{g}A^{ij}\right) + A^{jp}\left\{ \begin{matrix} i \\ j\ p \end{matrix} \right\}.
\end{aligned}$$

Example 24. If A^{ij} is a skew-symmetric tensor, prove that

$$A^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (A^{ij} \sqrt{g}) .$$

In the previous example we have proved

$$A^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (A^{ij} \sqrt{g}) + A^{jp} \left\{ \begin{matrix} i \\ j\ p \end{matrix} \right\} .$$

Now

$$A^{jp} \left\{ \begin{matrix} i \\ j\ p \end{matrix} \right\} = A^{pj} \left\{ \begin{matrix} i \\ p\ j \end{matrix} \right\}$$

(interchanging the dummy indices p and j)

$$= -A^{jp} \left\{ \begin{matrix} i \\ j\ p \end{matrix} \right\}$$

(since A^{ij} is skew symmetric and Christoffel symbol of second kind is symmetric in its last two indices).

Hence

$$A^{jp} \left\{ \begin{matrix} i \\ j\ p \end{matrix} \right\} = 0.$$

Therefore we get

$$A^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g} A^{ij}) .$$

Example 25. If A^{ij} is a tensor of type $(2, 0)$, prove that

$$A^{ij}_{,i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (A^{ij} \sqrt{g}) + A^{ir} \left\{ \begin{matrix} j \\ i\ r \end{matrix} \right\} .$$

We have

$$A^{ij}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + \left\{ \begin{matrix} i \\ k\ r \end{matrix} \right\} A^{rj} + \left\{ \begin{matrix} j \\ k\ r \end{matrix} \right\} A^{ir} .$$

Putting $k = i$ in the above equation we get

$$A^{ij}_{,i} = \frac{\partial A^{ij}}{\partial x^i} + \left\{ \begin{matrix} i \\ i\ r \end{matrix} \right\} A^{rj} + \left\{ \begin{matrix} j \\ i\ r \end{matrix} \right\} A^{ir}$$

$$= \frac{\partial A^{ij}}{\partial x^i} + A^{rj} \frac{\partial}{\partial x^r} (\log \sqrt{g}) + \left\{ \begin{matrix} j \\ i\ r \end{matrix} \right\} A^{ir}$$

$$= \frac{\partial A^{ij}}{\partial x^i} + A^{ij} \frac{\partial}{\partial x^i} (\log \sqrt{g}) + \left\{ \begin{matrix} j \\ i\ r \end{matrix} \right\} A^{ir}$$

(replacing the dummy index r by i in the second term)

$$= \frac{\partial A^{ij}}{\partial x^i} + A^{ij}\frac{\partial}{\partial x^i}\left(\frac{1}{2}\log g\right) + \left\{ \begin{matrix} j \\ i\ r \end{matrix} \right\} A^{ir}$$

$$= \frac{1}{\sqrt{g}}\left(\sqrt{g}\frac{\partial A^{ij}}{\partial x^i} + A^{ij}\frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x^i}\right) + \left\{ \begin{matrix} j \\ i\ r \end{matrix} \right\} A^{ir}$$

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^{ij}\right) + A^{ir}\left\{ \begin{matrix} j \\ i\ r \end{matrix} \right\}.$$

III.3.6 Covariant Derivative of a Mixed Tensor of Type (p, q)

Generalizing the previous results the formula for the covariant derivative of a tensor $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ of type (p, q) can be expressed as

$$\begin{aligned} A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k} &= \frac{\partial A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}}{\partial x^k} + \left\{ \begin{matrix} i_1 \\ k\ l \end{matrix} \right\} A^{l i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} + \left\{ \begin{matrix} i_2 \\ k\ l \end{matrix} \right\} A^{i_1 l i_3 \ldots i_p}_{j_1 j_2 \ldots j_q} \\ &+ \ldots + \left\{ \begin{matrix} i_p \\ k\ l \end{matrix} \right\} A^{i_1 i_2 \ldots i_{p-1} l}_{j_1 j_2 \ldots j_q} - \left\{ \begin{matrix} l \\ j_1\ k \end{matrix} \right\} A^{i_1 i_2 \ldots i_p}_{l j_2 \ldots j_q} \\ &- \left\{ \begin{matrix} l \\ j_2\ k \end{matrix} \right\} A^{i_1 i_2 \ldots i_p}_{j_1 l j_3 \ldots j_q} - \ldots - \left\{ \begin{matrix} l \\ j_q\ k \end{matrix} \right\} A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_{q-1} l}. \end{aligned} \tag{III.3.27}$$

It can be easily shown that $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k}$ are the components of a tensor of type $(p, q+1)$.

Note : The operation of covariant differentiation always produce another tensor, whose contravariant order is same as the parent tensor but the covariant order is increased by one. For this reason such type of differentiation is termed as covariant differentiation.

An important observation :

We know that an Euclidean space with rectangular Cartesian coordinate system is a particular type of Riemannian space where

$$g_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Due to this fact, in an Euclidean space with rectangular Cartesian coordinate system all the Christoffel symbols vanish identically. As a result in such an Euclidean space the covariant derivative of a tensor of any type reduces to ordinary partial derivative.

III.3.7 Ricci's Theorem :

Statement : In covariant differentiation the fundamental tensors g_{ij}, g^{ij} and the Kronecker delta δ^i_j behave like constants, *i.e.*,

$$g_{ij,k} = 0, \qquad g^{ij}_{,k} = 0 \text{ and } \delta^i_{j,k} = 0.$$

Proof : Applying the formula for covariant derivative of covariant tensor of order 2 we get

$$\begin{aligned} g_{ij,k} &= \frac{\partial g_{ij}}{\partial x^k} - \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} g_{lj} - \left\{ \begin{matrix} l \\ k\ j \end{matrix} \right\} g_{il} \\ &= \frac{\partial g_{ij}}{\partial x^k} - ([ik, j] + [kj, i]) \qquad \text{by (III.2.6)} \\ &= \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k}, \qquad \text{by (III.2.5)} \\ &= 0. \end{aligned} \tag{III.3.28}$$

Next, applying the formula for covariant derivative of contravariant tensors of order 2 we get

$$g^{ij}_{,k} = \frac{\partial g^{ij}}{\partial x^k} + \left\{ \begin{matrix} i \\ k\ l \end{matrix} \right\} g^{lj} + \left\{ \begin{matrix} j \\ k\ l \end{matrix} \right\} g^{il}. \tag{III.3.29}$$

We know

$$g_{ij} g^{jl} = \delta^l_i .$$

Therefore

$$\frac{\partial}{\partial x^k} (g_{ij} g^{jl}) = 0$$

or,

$$\frac{\partial g_{ij}}{\partial x^k} g^{jl} + g_{ij} \frac{\partial g^{jl}}{\partial x^k} = 0.$$

Multiplying both sides of the above relation by g^{im} we get

$$g^{im} g^{jl} \frac{\partial g_{ij}}{\partial x^k} + g^{im} g_{ij} \frac{\partial g^{jl}}{\partial x^k} = 0$$

or,

$$\delta^m_j \frac{\partial g^{jl}}{\partial x^k} = -g^{im} g^{jl} \frac{\partial g_{ij}}{\partial x^k},$$

or,

$$\begin{aligned} \frac{\partial g^{ml}}{\partial x^k} &= -g^{im} g^{jl} ([i\,k, j] + [j\,k, i]) \\ &= -g^{im} g^{jl} [i\,k, j] - g^{im} g^{jl} [j\,k, i] \\ &= -g^{im} \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} - g^{jl} \left\{ \begin{matrix} m \\ j\ k \end{matrix} \right\} \end{aligned}$$

or,

$$\frac{\partial g^{ij}}{\partial x^k} = -g^{li}\left\{\begin{matrix} j \\ l\ k \end{matrix}\right\} - g^{lj}\left\{\begin{matrix} i \\ j\ k \end{matrix}\right\}$$

(replacing the dummy indices i by l, m by i and l by j)

or,

$$\frac{\partial g^{ij}}{\partial x^k} + g^{il}\left\{\begin{matrix} j \\ l\ k \end{matrix}\right\} + g^{lj}\left\{\begin{matrix} i \\ j\ k \end{matrix}\right\} = 0$$

or,

$$g^{ij}_{,k} = 0. \tag{III.3.30}$$

Finally

$$\begin{aligned} \delta^i_{j,k} &= \frac{\partial \delta^i_j}{\partial x^k} + \delta^l_j\left\{\begin{matrix} i \\ l\ k \end{matrix}\right\} - \delta^i_l\left\{\begin{matrix} l \\ j\ k \end{matrix}\right\}. \\ &= 0 + \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\} - \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\} \\ &= 0. \end{aligned} \tag{III.3.31}$$

This completes the proof of this theorem.

Note : The results (III.3.28) and (III.3.30) are also known as Ricci's lemma.

Example 26. If A_i is a covariant vector, prove that $\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)$ is a covariant tensor of rank 2.

We know

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \left\{\begin{matrix} l \\ i\ j \end{matrix}\right\} A_l.$$

So

$$\frac{\partial A_i}{\partial x^j} = A_{i,j} + \left\{\begin{matrix} l \\ i\ j \end{matrix}\right\} A_l.$$

Now

$$\begin{aligned} \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} &= A_{i,j} + \left\{\begin{matrix} l \\ i\ j \end{matrix}\right\} A_l - A_{j,i} - \left\{\begin{matrix} l \\ j\ i \end{matrix}\right\} A_l \\ &= A_{i,j} - A_{j,i}. \end{aligned}$$

Since the covariant derivative of a covariant vector is a covariant tensor of order 2, from the above equation it follows that $A_{i,j} - A_{j,i}$ is a covariant tensor of rank 2.

Example 27. If a_{ij} is a symmetric non-singular tensor ($|a_{ij}| \neq 0$) of type $(0, 2)$, such that $a_{ij,k} = 0$, prove that

$$\left\{\begin{matrix} l \\ i\ j \end{matrix}\right\} = \frac{1}{2}a^{lk}\left(\frac{\partial a_{ik}}{\partial x^j} + \frac{\partial a_{jk}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^k}\right).$$

Since $|a_{ij}| \neq 0$, the reciprocal a^{ij} of a_{ij} is given by

$$a^{ij} = \frac{\text{cofactor of } a_{ij} \text{ in } |a_{ij}|}{|a_{ij}|}. \tag{III.3.32}$$

By proposition

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} a_{lj} - \left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\} a_{il} = 0.$$

So

$$\frac{\partial a_{ij}}{\partial x^k} = \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} a_{lj} + \left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\} a_{il}. \tag{III.3.33}$$

Similarly we can get

$$\frac{\partial a_{jk}}{\partial x^i} = \left\{ \begin{matrix} l \\ j\ i \end{matrix} \right\} a_{lk} + \left\{ \begin{matrix} l \\ k\ i \end{matrix} \right\} a_{jl} \tag{III.3.34}$$

and

$$\frac{\partial a_{ki}}{\partial x^j} = \left\{ \begin{matrix} l \\ k\ j \end{matrix} \right\} a_{li} + \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} a_{kl}. \tag{III.3.35}$$

Adding (III.3.34) and (III.3.35) and subtracting (III.3.33) from the sum we get

$$\begin{aligned} \frac{\partial a_{ik}}{\partial x^j} + \frac{\partial a_{jk}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^k} &= \left\{ \begin{matrix} l \\ j\ i \end{matrix} \right\} a_{lk} + \left\{ \begin{matrix} l \\ k\ i \end{matrix} \right\} a_{jl} + \left\{ \begin{matrix} l \\ k\ j \end{matrix} \right\} a_{li} \\ &\quad + \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} a_{kl} - \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} a_{lj} - \left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\} a_{il} \\ &= 2 \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} a_{kl} \end{aligned}$$

(since a_{ij} is symmetric and Christoffel symbol of second kind is symmetric in its last two indices).

$$= 2 \left\{ \begin{matrix} m \\ i\ j \end{matrix} \right\} a_{km}$$

(replacing the dummy index l by m).

Hence we get

$$\begin{aligned} \frac{1}{2} a^{lk} \left(\frac{\partial a_{ik}}{\partial x^j} + \frac{\partial a_{jk}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^k} \right) &= \left\{ \begin{matrix} m \\ i\ j \end{matrix} \right\} a^{lk} a_{km} \\ &= \left\{ \begin{matrix} m \\ i\ j \end{matrix} \right\} \delta^l_m = \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\}. \end{aligned}$$

Note : It can be easily shown that

i) the covariant differentiation of sum or difference of any two tensors of same type obey the same rules like ordinary differentiation,

ii) the covariant differentiation of product of any two tensors obey the same rule of ordinary differentiation.

It will be clear from the following example.

Example 28. Let A_{ij} and B_{ij} be any two covariant tensors of rank 2, then show that

$$\text{(i)}\quad (A_{ij}+B_{ij})_{,k} = A_{ij,k}+B_{ij,k},$$

$$\text{(ii)}\quad (A_{ij}-B_{ij})_{,k} = A_{ij,k}-B_{ij,k}$$

and

$$\text{(iii)}\quad (A_{ij}B_{lm})_{,k} = A_{ij,k}B_{lm}+A_{ij}B_{lm,k}.$$

Let $C_{ij} = A_{ij}+B_{ij}$ for all $i,j = 1,2,\ldots,n$. Then C_{ij} is a tensor of type $(0,2)$. Now

$$\begin{aligned}
(A_{ij}+B_{ij})_{,k} &= C_{ij,k}\\
&= \frac{\partial C_{ij}}{\partial x^k} - \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} C_{lj} - \left\{ \begin{matrix} l \\ k\ j \end{matrix} \right\} C_{il}\\
&= \frac{\partial}{\partial x^k}(A_{ij}+B_{ij}) - \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\}(A_{lj}+B_{lj}) - \left\{ \begin{matrix} l \\ k\ j \end{matrix} \right\}(A_{il}+B_{il})\\
&= \frac{\partial A_{ij}}{\partial x^k} - \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} A_{lj} - \left\{ \begin{matrix} l \\ k\ j \end{matrix} \right\} A_{il}\\
&\quad + \frac{\partial B_{ij}}{\partial x^k} - \left\{ \begin{matrix} l \\ i\ k \end{matrix} \right\} B_{lj} - \left\{ \begin{matrix} l \\ k\ j \end{matrix} \right\} B_{il}\\
&= A_{ij,k}+B_{ij,k}. \qquad \text{(III.3.36)}
\end{aligned}$$

Similarly it can be proved that

$$(A_{ij}-B_{ij})_{,k} = A_{ij,k}-B_{ij,k}. \qquad \text{(III.3.37)}$$

Next let

$$D_{ijlm} = A_{ij}B_{lm} \qquad (i,j,l,m = 1,2,\ldots,n). \qquad \text{(III.3.38)}$$

So, D_{ijlm} is a tensor of type $(0,4)$. Now

$$\begin{aligned}
(A_{ij}B_{lm})_{,k} &= D_{ijlm,k}\\
&= \frac{\partial D_{ijlm}}{\partial x^k} - \left\{ \begin{matrix} h \\ k\ i \end{matrix} \right\} D_{hjlm} - \left\{ \begin{matrix} h \\ k\ j \end{matrix} \right\} D_{ihlm}\\
&\quad - \left\{ \begin{matrix} h \\ k\ l \end{matrix} \right\} D_{ijhm} - \left\{ \begin{matrix} h \\ k\ m \end{matrix} \right\} D_{ijlh}\\
&= \frac{\partial}{\partial x^k}(A_{ij}B_{lm}) - \left\{ \begin{matrix} h \\ k\ i \end{matrix} \right\} A_{hj}B_{lm} - \left\{ \begin{matrix} h \\ k\ j \end{matrix} \right\} A_{ih}B_{lm}\\
&\quad - \left\{ \begin{matrix} h \\ k\ l \end{matrix} \right\} A_{ij}B_{hm} - \left\{ \begin{matrix} h \\ k\ m \end{matrix} \right\} A_{ij}B_{lh}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial A_{ij}}{\partial x^k} - \left\{ \begin{matrix} h \\ k\ i \end{matrix} \right\} A_{hj} - \left\{ \begin{matrix} h \\ k\ j \end{matrix} \right\} A_{ih}\right) B_{lm} \\
&\quad + A_{ij}\left(\frac{\partial B_{lm}}{\partial x^k} - \left\{ \begin{matrix} h \\ k\ l \end{matrix} \right\} B_{hm} - \left\{ \begin{matrix} h \\ k\ m \end{matrix} \right\} B_{lh}\right) \\
&= A_{ij,k}B_{lm} + A_{ij}B_{lm,k}. \qquad \text{(III.3.39)}
\end{aligned}
$$

Example 29. If A^i and B^i are two unit contravariant vectors, show that they are inclined to each other at a constant angle if and only if

$$A^i_{,k}B_i + B^i_{,k}A_i = 0.$$

Since A^i and B^i are unit contravariant vectors we have

$$
\left.
\begin{aligned}
A^iA_i &= g_{ij}A^iA^j = 1 \\
&\text{and} \\
B^iB_i &= g_{ij}B^iB^j = 1
\end{aligned}
\right\}. \qquad \text{(III.3.40)}
$$

The angle θ between A^i and B^i is given by

$$
\begin{aligned}
\cos\theta &= \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}} \\
&= g_{ij}A^iB^j. \qquad \text{(III.3.41)}
\end{aligned}
$$

Now, the angle θ between A^i and B^i is constant if and only if the covariant derivative of the right hand side of (III.3.41) is zero (since $\cos\theta$ is an invariant its covariant derivative is nothing but its ordinary partial derivative).
Now

$$
\begin{aligned}
\left(g_{ij}A^iB^j\right)_{,k} &= g_{ij,k}A^iB^j + g_{ij}A^i_{,k}B^j + g_{ij}A^iB^j_{,k} \\
&= A^i_{,k}g_{ij}B^j + g_{ij}B^j_{,k}A^i \\
&= A^i_{,k}B_i + B^i_{,k}A_i.
\end{aligned}
$$

Hence the result follows.

Example 30. For any contravariant vector A^i, show that

$$\left(g_{ij}A^i\right)_{,k} = A_{j,k}\ .$$

We have

$$
\begin{aligned}
\left(g_{ij}A^i\right)_{,k} &= g_{ij,k}A^i + g_{ij}A^i_{,k} \\
&= g_{ij}A^i_{,k} \qquad \text{(applying Ricci's theorem)}
\end{aligned}
$$

$$
\begin{aligned}
&= g_{ij}\left(\frac{\partial A^i}{\partial x^k} + \left\{ \begin{matrix} i \\ l\ k \end{matrix} \right\} A^l\right) \qquad \text{using (III.3.9)} \\
&= g_{ij}\frac{\partial}{\partial x^k}\left(g^{il}A_l\right) + g_{ij}\left\{ \begin{matrix} i \\ l\ k \end{matrix} \right\} A^l \\
&= g_{ij}\frac{\partial g^{il}}{\partial x^k}A_l + g_{ij}g^{il}\frac{\partial A_l}{\partial x^k} + [lk,j]\,A^l \quad \text{by (III.2.6)} \\
&= g_{ij}A_l\left(-g^{lm}\left\{ \begin{matrix} i \\ m\ k \end{matrix} \right\} - g^{im}\left\{ \begin{matrix} l \\ m\ k \end{matrix} \right\}\right) + \delta^l_j\frac{\partial A_l}{\partial x^k} + [lk,j]\,A^l \\
&\qquad\qquad \text{using (III.2.9)} \\
&= -g^{lm}A_l g_{ij}\left\{ \begin{matrix} i \\ m\ k \end{matrix} \right\} - g_{ij}g^{im}A_l\left\{ \begin{matrix} l \\ m\ k \end{matrix} \right\} + \frac{\partial A_j}{\partial x^k} + [lk,j]\,A^l \\
&= -A^m\,[mk,j] - \delta^m_j A_l\left\{ \begin{matrix} l \\ m\ k \end{matrix} \right\} + \frac{\partial A_j}{\partial x^k} + [mk,j]\,A^m \\
&\qquad \text{(replacing the dummy index } l \text{ by } m \text{ in the last element)} \\
&= -A_l\left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\} + \frac{\partial A_j}{\partial x^k} \\
&= A_{j,k}.
\end{aligned}
$$

III.3.8 Gradient of a Scalar

Let f be any scalar. We have already proved that the partial derivative

$$f_{,i} = \frac{\partial f}{\partial x^i} \tag{III.3.42}$$

are the components of a covariant vector. This is called the gradient of f and is denoted by ∇f or grad f.

III.3.9 Divergence of a Contravariant Vector

Let us consider an arbitrary contravariant vector λ^i. Then the covariant derivative $\lambda^i_{,j}$ of λ^i (with respect to the fundamental metric tensor g_{ij}) is a tensor of type $(1,1)$. Contraction of the contravariant and covariant indices yields an invariant, which is called divergence of the vector λ^i and is denoted by $\mathrm{div}\lambda^i$. Thus

$$
\begin{aligned}
\mathrm{div}\,\lambda^i &= \lambda^i_{,i} \\
&= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^l}\left(\sqrt{g}\lambda^l\right).
\end{aligned} \tag{III.3.43}
$$

III.3.10 Divergence of a Covariant Vector

Let us consider an arbitrary covariant vector μ_i. Then $\mu^i = g^{ij}\mu_j$ is a contravariant vector. The divergence of the contravariant vector μ^i is said to be the divergence of μ_i.

Therefore

$$\begin{aligned} \operatorname{div} \mu_i &= \operatorname{div} \mu^i \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^l} \left(\sqrt{g} \mu^l\right). \end{aligned} \tag{III.3.44}$$

Example 31. If the components A_1, A_2, A_3 of a vector in cylindrical coordinates ρ, ϕ, z are $\rho, z \sin \phi$ and $e^{\phi} \cos z$, prove that

$$\operatorname{div} A_i = 2 + \frac{z}{\rho^2} \cos \phi - e^{\phi} \sin z.$$

We know, in cylindrical coordinates (ρ, ϕ, z) the square of the line element is given by (see equation (III.1.10) of example 3 on Page 70)

$$ds^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2.$$

Then

$$g_{11} = 1, \; g_{22} = \rho^2, \; g_{33} = 1 \text{ and } g_{ij} = 0 \text{ if } i \neq j.$$

So

$$g = \det g_{ij} = \rho^2 .$$

Hence we get

$$g^{11} = 1, \; g^{22} = \frac{1}{\rho^2}, \; g^{33} = 1 \text{ and } g^{ij} = 0 \text{ if } i \neq j.$$

And

$$\begin{aligned} A^1 &= g^{1i} A_i = g^{11} A_1 = A_1, \\ A^2 &= g^{2i} A_i = g^{22} A_2 = \tfrac{1}{\rho^2} z \sin \phi \end{aligned}$$

and

$$A^3 = g^{3i} A_i = g^{33} A_3 = e^{\phi} \cos z.$$

Therefore

$$\begin{aligned} \operatorname{div} A_i &= \operatorname{div} A^i \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^l} \left(\sqrt{g} A^l\right) \\ &\qquad (\text{here } x^1 = \rho, \; x^2 = \phi, \; x^3 = z) \\ &= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho^2) + \frac{\partial}{\partial \phi} \left(\rho \frac{1}{\rho^2} z \sin \phi \right) + \frac{\partial}{\partial z} \left(\rho e^{\phi} \cos z \right) \right] \\ &\qquad (\text{since } \sqrt{g} = \rho) \\ &= \frac{1}{\rho} \left[2\rho + \frac{1}{\rho} z \cos \phi + \rho e^{\phi} (-\sin z) \right] \\ &= 2 + \frac{1}{\rho^2} z \cos \phi - e^{\phi} \sin z. \end{aligned}$$

Note : This example shows the utility of the formula (III.3.44) that we need not calculate the Christoffel symbols.

III.3.11 Conservative Vector

If the divergence of a vector (covariant or contravariant) vanishes identically, then the vector is said to be conservative.

Example 32. Prove that in V_2 with line element

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2,$$

the covariant vector with components $x^1 \cos 2x^2, -(x^1)^2 \sin 2x^2$ is conservative.

Here

$$g_{11} = 1,\ g_{22} = (x^1)^2,\ \text{and}\ g_{12} = g_{21} = 0.$$

Hence

$$g = \det g_{ij} = (x^1)^2 .$$

Now

$$g^{11} = 1,\ g^{22} = \frac{1}{(x^1)^2},\ g^{21} = g^{12} = 0.$$

Then

$$\begin{aligned} A^1 &= g^{1j}A_j = g^{11}A_1 = x^1 \cos 2x^2, \\ A^2 &= g^{2j}A_j = g^{22}A_2 = \tfrac{1}{(x^1)^2}\left[-(x^1)^2 \sin 2x^2\right] = -\sin 2x^2. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{div} A_i &= \operatorname{div} A^i \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^l} (\sqrt{g} A^l) \\ &= \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial x^1} (x^1 x^1 \cos 2x^2) + \frac{\partial}{\partial x^2} (x^1 (-\sin 2x^2)) \right] \\ &= \frac{1}{\sqrt{g}} \left[2x^1 \cos 2x^2 - 2x^1 \cos 2x^2 \right] \\ &= 0. \end{aligned}$$

Therefore the vector A_i is conservative.

III.3.12 Divergence of a Contravariant Tensor of Order Two

Let α^{ij} be an arbitrary contravariant tensor of order 2. The divergence of α^{ij} is defined by

$$\operatorname{div} \alpha^{ij} = \alpha^{ij}_{,j} . \qquad \text{(III.3.45)}$$

We know

$$\alpha^{ij}_{,k} = \frac{\partial \alpha^{ij}}{\partial x^k} + \alpha^{lj} \left\{ \begin{matrix} i \\ l\ k \end{matrix} \right\} + \alpha^{il} \left\{ \begin{matrix} j \\ l\ k \end{matrix} \right\}.$$

Putting $k = j$ we get from the above equation

$$\begin{aligned}
\alpha^{ij}_{,j} &= \frac{\partial \alpha^{ij}}{\partial x^j} + \alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} + \alpha^{il}\left\{\begin{matrix} j \\ l\ j \end{matrix}\right\} \\
&= \frac{\partial \alpha^{ij}}{\partial x^j} + \alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} + \alpha^{il}\frac{\partial}{\partial x^l}(\log\sqrt{g}) \\
&= \alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} + \frac{\partial \alpha^{ij}}{\partial x^j} + \alpha^{ij}\frac{\partial}{\partial x^j}(\log\sqrt{g}) \\
&\qquad \text{(replacing the dummy index } j \text{ by } l \text{ in the last term)} \\
&= \alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} + \frac{\partial \alpha^{ij}}{\partial x^j} + \frac{1}{\sqrt{g}}\alpha^{ij}\frac{\partial}{\partial x^j}(\sqrt{g}) \\
&= \alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} + \frac{1}{\sqrt{g}}\left(\sqrt{g}\frac{\partial \alpha^{ij}}{\partial x^j} + \alpha^{ij}\frac{\partial}{\partial x^j}(\sqrt{g})\right).
\end{aligned}$$

Therefore

$$\operatorname{div}\alpha^{ij} = \alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\alpha^{ij}). \qquad \text{(III.3.46)}$$

Example 33. If α^{ij} is a skew-symmetric tensor,

$$\operatorname{div}\alpha^{ij} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\alpha^{ij}).$$

We have already proved

$$\operatorname{div}\alpha^{ij} = \alpha^{ij}_{,j} = \alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\alpha^{ij}).$$

Now

$$\begin{aligned}
\alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} &= \alpha^{jl}\left\{\begin{matrix} i \\ j\ l \end{matrix}\right\} \\
&\qquad \text{(interchanging the dummy indices } j \text{ and } l) \\
&= -\alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} \\
&\qquad \text{(since } \alpha^{jl} \text{ is skew-symmetric).}
\end{aligned}$$

This implies

$$\alpha^{lj}\left\{\begin{matrix} i \\ l\ j \end{matrix}\right\} = 0.$$

Hence we get

$$\operatorname{div}\alpha^{ij} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\alpha^{ij}).$$

III.3.13 Divergence of a Mixed Tensor of Type $(1,1)$

Let α^i_j be an arbitrary tensor of type $(1,1)$. Then the divergence of α^i_j is defined by

$$\operatorname{div} \alpha^i_j = \alpha^i_{j,i} \,. \tag{III.3.47}$$

We know

$$\alpha^i_{j,k} = \frac{\partial \alpha^i_j}{\partial x^k} + \left\{ \begin{matrix} i \\ l\ k \end{matrix} \right\} \alpha^l_j - \left\{ \begin{matrix} l \\ j\ k \end{matrix} \right\} \alpha^i_l \,.$$

Putting $k = i$, we get from the above

$$\begin{aligned}
\alpha^i_{j,i} &= \frac{\partial \alpha^i_j}{\partial x^i} + \left\{ \begin{matrix} i \\ l\ i \end{matrix} \right\} \alpha^l_j - \left\{ \begin{matrix} l \\ j\ i \end{matrix} \right\} \alpha^i_l \\
&= \frac{\partial \alpha^i_j}{\partial x^i} + \alpha^l_j \frac{\partial}{\partial x^l} (\log \sqrt{g}) - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} \alpha^i_l \\
&= \frac{\partial \alpha^i_j}{\partial x^i} + \alpha^i_j \frac{\partial}{\partial x^i} (\log \sqrt{g}) - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} \alpha^i_l \\
&\qquad \text{(replacing the dummy index } l \text{ by } i \text{ in the second term)} \\
&= \frac{1}{\sqrt{g}} \left(\sqrt{g} \frac{\partial \alpha^i_j}{\partial x^i} + \alpha^i_j \frac{\partial}{\partial x^i} (\sqrt{g}) \right) - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} \alpha^i_l \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} \alpha^i_j) - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} \alpha^i_l \,.
\end{aligned} \tag{III.3.48}$$

Example 34. If the associate tensor α^{ij} of the tensor α^i_j of type $(1,1)$ is symmetric,

$$\begin{aligned}
\alpha^i_{j,i} &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\alpha^i_j \sqrt{g}) - \frac{1}{2} \alpha^{ik} \frac{\partial g_{ik}}{\partial x^j} \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\alpha^i_j \sqrt{g}) - \frac{1}{2} \alpha_{ik} \frac{\partial g^{ik}}{\partial x^j} \,.
\end{aligned}$$

We have already proved for any tensor α^i_j

$$\begin{aligned}
\alpha^i_{j,i} &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} \alpha^i_j) - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} \alpha^i_l \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} \alpha^i_j) - \alpha^i_l g^{lk} [ij, k] \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} \alpha^i_j) - \alpha^{ik} \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} \alpha^i_j) - \frac{1}{2} \alpha^{ik} \frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2} \alpha^{ik} \frac{\partial g_{jk}}{\partial x^i} + \frac{1}{2} \alpha^{ik} \frac{\partial g_{ij}}{\partial x^k} \,.
\end{aligned} \tag{III.3.49}$$

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\alpha^i_j) - \frac{1}{2}\alpha^{ik}\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2}\alpha^{ik}\frac{\partial g_{jk}}{\partial x^i} + \frac{1}{2}\alpha^{ki}\frac{\partial g_{kj}}{\partial x^i}$$

(interchanging the dummy indices i and k in the last term)

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\alpha^i_j) - \frac{1}{2}\alpha^{ik}\frac{\partial g_{ik}}{\partial x^j} \qquad (\text{since } \alpha^{ik} = \alpha^{ki}).$$

The other result can be proved similarly.

III.3.14 Laplacian of an Invariant

For an invariant f, $f_i = \frac{\partial f}{\partial x^i}$ is a covariant vector. Hence

$$f^j = g^{ij} f_i$$

is a contravariant vector. Then

$$\text{div}\, f^i = f^i_{,i} = (g^{ji} f_j)_{,i} \qquad \text{(III.3.50)}$$

is an invariant and is called Laplacian of f. This is denoted by $\nabla^2 f$. Thus

$$\begin{aligned} \nabla^2 f &= \text{div}\, f^i \\ &= \text{div}\, f_i \\ &= \text{div}\,(grad\, f). \end{aligned} \qquad \text{(III.3.51)}$$

An Expression for the Laplacian of an Invariant

We have

$$\begin{aligned} f^{ij}_{,i} &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g} f^k) \\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g} g^{kj} f_j) \\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g} g^{kj}\frac{\partial f}{\partial x^j}\right). \end{aligned} \qquad \text{(III.3.52)}$$

III.3.15 Curl of a Covariant Vector

Let A_i be a covariant vector.Then

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} A_l .$$

So

$$\frac{\partial A_i}{\partial x^j} = A_{i,j} + \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} A_l .$$

Therefore

$$\begin{aligned}\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} &= A_{i,j} + \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} A_l - A_{j,i} - \left\{ \begin{matrix} l \\ j\ i \end{matrix} \right\} A_l \\ &= A_{i,j} - A_{j,i}.\end{aligned}$$

This shows that $\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$ is a covariant tensor of rank 2.
This tensor is called the curl of the covariant vector.
Thus

$$\begin{aligned}\text{curl } A_i &= A_{i,j} - A_{j,i} \\ &= \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}.\end{aligned} \tag{III.3.53}$$

Curl of a vector A_i is also called rotation of the vector A_i.
From (III.3.53) it follows that curl of a vector is skew-symmetric.

Irrotational Vector

If the curl of a covariant vector vanishes identically, then the vector is said to be irrotational.

If λ_i is an irrotational vector, then

$$\text{curl}\, \lambda_i = 0$$

and hence

$$\frac{\partial \lambda_i}{\partial x^j} = \frac{\partial \lambda_j}{\partial x^i}.$$

Example 35. For an invariant f, curl (grad f) = 0.

Let f be an invariant. Then

$$\text{grad}\, f = f_i = \frac{\partial f}{\partial x^i}$$

are the components of a covariant vector.
Now

$$\begin{aligned}\text{curl}\, f_i &= \frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i} \\ &= \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right) - \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) \\ &= \frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \\ &= 0.\end{aligned}$$

Hence curl (grad f) = 0.

Example 36. If curl of a covariant vector λ_i vanishes identically, then λ_i is gradient.

Given curl $\lambda_i = 0$. This implies

$$\frac{\partial \lambda_i}{\partial x^j} = \frac{\partial \lambda_j}{\partial x^i}$$

and hence $\lambda_i dx^i$ is an exact differential.
Let $\lambda_i dx^i = df$, where f is an invariant.
Now

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

So

$$\left(\lambda_i - \frac{\partial f}{\partial x^i}\right) dx^i = 0.$$

Since dx^i is an arbitrary contravariant vector it follows that

$$\lambda_i = \frac{\partial f}{\partial x^i} = f_{,i}.$$

In other words, λ_i is gradient.

Example 37. If f is an invariant, then

$$f_{,ij} = f_{,ji}.$$

Since f is an invariant, $f_{,i} = \frac{\partial f}{\partial x^i}$ is a covariant vector. Now $f_{,ij}$ is the repeated covariant derivative of f. Thus

$$\begin{aligned}
f_{,ij} &= f_{,i,j} \\
&= \frac{\partial f_i}{\partial x^j} - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} f_l \\
&= \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right) - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} f_l \\
&= \frac{\partial^2 f}{\partial x^j \partial x^i} - \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} f_l \\
&= \frac{\partial^2 f}{\partial x^i \partial x^j} - \left\{ \begin{matrix} l \\ j\ i \end{matrix} \right\} f_l \\
&\qquad \text{(using Young's theorem for calculus of several variables)} \\
&= f_{,ji}.
\end{aligned}$$